2021 - ISU Putnam Practice Set 4 - Solutions

Friday, October 27, 2022

Matrices

1. Do there exist $n \times n$ matrices A and B such that $AB - BA = I_n$?

   Solution: The answer is negative. The trace of $AB - BA$ is zero since $\text{tr}(AB) = \text{tr}(BA)$, while the trace of I_n is n; the matrices cannot be equal.

2. Let Abe the $n \times n$ matrix whose i, j entry is $i + j$ for all $i, j = 1, 2, ..., n$. What is the rank of A?

   Solution: If $n = 1$, then $\text{rank}(A) = 1$. If $n \geq 2$, then A is row-equivalent to the following matrix (and thus has the same rank:

$$\begin{pmatrix} 2 & 3 & 4 & \cdots & n \\ 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

   Thus $\text{rank}(A) = 2$.

3. Let A and B be $n \times n$ matrices, $n \geq 1$, satisfying $AB - B^2A^2 = I_n$ and $A^3 + B^3 = 0$. Prove that $BA - A^2B^2 = I_n$.

   Solution: We have

$$(A + iB^2)(B + iA^2) = AB - B^2A^2 + i(A^3 + B^3) = I_n.$$

   This $A + iB^2$ is invertible, and its inverse is $B + iA^2$. Then

$$I_n = (B + iA^2)(A + iB^2) = BA - A^2B^2 + i(A^3 + B^3) = BA - A^2B^2$$

   as desired.

4. Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible?

Solution Putnam 1991 A2:

$$(M^2+N^2)M = M^3+N^2M = N^3+M^2N = (M^2+N^2)N.$$

But now if M^2+N^2 was invertible, we could multiply by its inverse to get $M=N$, which is a contradiction. So the answer is no.

5. Let V be an infinite set of vectors in $\mathbb{R}^n$ containing n linearly independent vectors. A finite subset $S \subseteq V$ is called crucial if the set $V \setminus S$ contains no n linearly independent vectors, but every set $V \setminus T$, with T a proper subset of S does. Prove there are only finitely many crucial subsets of V .

   Solution: Let S be a crucial subset of V. Let V_S be the vector space spanned by $V \setminus S$. By adding any vector of S to V_S, we turn this space into $\mathbb{R}^n$. This implies that V_S is a vector space of dimension $n-1$ and all the vectors of V but the ones in S are in V_S. Now let $W = \bigcap V_S$, where the intersection is over all crucial subsets. By the finiteness of dimensions, W can be written as the intersection of a finite collection of spaces V_S, say $W = V_{S_1} \cap V_{S_2} \cap \cdots \cap V_{S_m}$, and assume m is minimal. Starting with V_{S_1} we add at each step to the intersection $V_{S_1} \cap V_{S_2} \cap \cdots \cap V_{S_j}$ a subspace $V_{S_{j+1}}$ such that

$$\dim(V_{S_1} \cap V_{S_2} \cap \cdots \cap V_{S_j}) > \dim(V_{S_1} \cap V_{S_2} \cap \cdots \cap V_{S_j} \cap V_{S_{j+1}}).$$

   Hence $m \leq n$. As all but finitely many vectors of V belong to V_{S_i} , we conclude that all but finitely many vectors of V belong to W. But the vectors from S do not belong to V_S, and hence do not belong to W, for any crucial subset S. Then we only have finitely many vectors of V, namely those not in W, to choose from for building a crucial set. Thus there are only finitely many crucial sets.