2021 - ISU Putnam Practice Set 4 - Solutions

Wednesday, September 29, 2021

More Calculus

1. A function $f : D \to \mathbb{R}$ (where D is an interval) is called convex (you might know this better as concave-up) if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $0 < \lambda < 1$ and all $x, y \in D$. If $-f$ is convex, then f is called concave.

Show that if a function $f : \mathbb{R} \to \mathbb{R}$ is both convex and concave, then $f(x) = ax + b$ for some constants $a, b \in \mathbb{R}$.

Solution: Set $b = f(0)$ and $a = f(1) - b$. Let $0 < x < 1$. Then setting $x = 1, y = 0, \lambda = x$ above, we get

$$f(x) = x(a+b) + (1-x)b = ax + b.$$

A similar argument works for $x > 1$ and $x < 0$.

2. Show that

$$\sqrt[3]{3+\sqrt[3]{3}} + \sqrt[3]{3-\sqrt[3]{3}} < 2\sqrt[3]{3}.$$

Solution: Set $f(x) = \sqrt[3]{x}$. As $f''(x) = -\frac{2}{9}x^{-5/3} < 0$ for $x > 0$, f is concave on $(0, \infty)$. Therefore $\frac{f(a)+f(b)}{2} < f\left(\frac{a+b}{2}\right)$ for all $0 < a < b$. Setting $a = 3 - \sqrt[3]{3}$ and $b = 3 + \sqrt[3]{3}$, we get

$$\frac{1}{2}\left(\sqrt[3]{3+\sqrt[3]{3}} + \sqrt[3]{3-\sqrt[3]{3}}\right) < \sqrt[3]{3}$$

from which the inequality follows.

3. Show that

$$(\sin x)^{\sin x} < (\cos x)^{\cos x}$$

for all $0 < x < \frac{\pi}{4}$.

Solution: Since $0 < \sin(x) < \frac{\sqrt{2}}{2} < \cos(x) < 1$ for all $0 < x < \frac{\pi}{4}$, it would suffice to show that the function $f(x) = x^x$ is increasing - except this is not true!

Instead, since $\ln(x)$ is increasing, the above inequality is equivalent to showing that

$$\sin(x)\ln(\sin x) < \cos(x)\ln(\cos x).$$

Since $\sin(x) > 0$ on $(0, \frac{\pi}{4})$, this is equivalent to

$$\ln(\cos x) - \tan(x)\ln(\sin x) > 0.$$

The function $\ln(x)$ is concave (check it!) and so

$$\lambda \ln(a) + (1-\lambda)\ln(b) < \ln(\lambda a(1-\lambda)b)$$

for all $a, b \in (0, \infty)$. Taking $a = \sin(x)$ and $b = \sin(x) + \cos(x)$ and $\lambda = \tan(x)$, we get

$$\begin{aligned}\tan(x)\ln(\sin(x)) + (1-\tan(x))\ln(\sin(x)+\cos(x)) &< \ln(\tan(x)\sin(x) + (1-\tan(x))(\sin(x)+\cos(x))) \\ &= \ln(\cos(x)).\end{aligned}$$

We are done by noticing that $(1-\tan(x))\ln(\sin(x)+\cos(x)) > 0$ since $\sin(x)+\cos(x) = \sqrt{2}\cos(\pi/4 - x) > 1$.

4. Let k be a fixed positive integer. The n-th derivative of $\frac{1}{x^k-1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

Solution: By differentiating $P_n(x)/(x^k-1)^{n+1}$, we find that $P_{n+1}(x) = (x^k-1)P_n'(x) - (n+1)kx^{k-1}P_n(x)$; substituting $x = 1$ yields $P_{n+1}(1) = -(n+1)kP_n(1)$. Since $P_0(1) = 1$, an easy induction gives $P_n(1) = (-k)^n n!$ for all $n \geq 0$.

Note: one can also argue by expanding in Taylor series around 1. Namely, we have

$$\frac{1}{x^k-1} = \frac{1}{k(x-1)+\cdots} = \frac{1}{k}(x-1)^{-1} + \cdots,$$

so

$$\frac{d^n}{dx^n}\frac{1}{x^k-1} = \frac{(-1)^n n!}{k(x-1)^{-n-1}}$$

and

$$\begin{aligned}P_n(x) &= (x^k-1)^{n+1}\frac{d^n}{dx^n}\frac{1}{x^k-1} \\ &= (k(x-1)+\cdots)^{n+1}\left(\frac{(-1)^n n!}{k}(x-1)^{-n-1} + \cdots\right) \\ &= (-k)^n n! + \cdots.\end{aligned}$$

5. Let $f : [a,b] \to \mathbb{R}$ be a convex function. Show that f is continuous on (a,b).

   Solution: Fix $a < x_0 < b$. Let α and β be two limit points of f at x_0: α from the left and β from the right. We want to prove that they are equal. If not, without loss of generality we can assume that $\alpha < \beta$. Choose $x < x_0$ and $y > x_0$ such that $|f(x) - \alpha|$ and $|f(y) - \beta|$ are small. Because β is a limit point, there exist points on the graph of f close to (x_0, β) and hence above the line segment from $(x, f(x))$ to $(y, f(y))$. This contradicts the convexity of f. Hence $\alpha = \beta$. Hence f has a limit at x_0. Redo the argument for $x = x_0$ to conclude that the limit is equal to $f(x_0)$. Hence f is continuous at x_0.