More Calculus

1. A function \( f : D \to \mathbb{R} \) (where \( D \) is an interval) is called convex (you might know this better as concave-up) if
\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)
\]
for all \( 0 < \lambda < 1 \) and all \( x, y \in D \). If \( -f \) is convex, then \( f \) is called concave.
Show that if a function \( f : \mathbb{R} \to \mathbb{R} \) is both convex and concave, then \( f(x) = ax + b \) for some constants \( a, b \in \mathbb{R} \).

**Solution:** Set \( b = f(0) \) and \( a = f(1) - b \). Let \( 0 < x < 1 \). Then setting \( x = 1, y = 0, \lambda = x \) above, we get
\[
f(x) = x(a + b) + (1 - x)b = ax + b.
\]
A similar argument works for \( x > 1 \) and \( x < 0 \).

2. Show that
\[
3\sqrt{3} + \sqrt{3} + 3\sqrt{3 - 3\sqrt{3}} < 2\sqrt{3}.
\]

**Solution:** Set \( f(x) = \sqrt[3]{x} \). As \( f''(x) = -\frac{2}{9}x^{-5/3} < 0 \) for \( x > 0 \), \( f \) is concave on \((0, \infty)\).
Therefore \( \frac{f(a) + f(b)}{2} < f \left( \frac{a+b}{2} \right) \) for all \( 0 < a < b \). Setting \( a = 3 - \sqrt{3} \) and \( b = 3 + \sqrt{3} \), we get
\[
\frac{1}{2} \left( 3\sqrt{3} + \sqrt{3} + \sqrt{3 - \sqrt{3}} \right) < 3\sqrt{3}
\]
from which the inequality follows.

3. Show that
\[
(sin x)^{sin x} < (cos x)^{cos x}
\]
for all \( 0 < x < \frac{\pi}{4} \).

**Solution:** Since \( 0 < \sin(x) < \sqrt{2} / 2 < \cos(x) < 1 \) for all \( 0 < x < \frac{\pi}{4} \), it would suffice to show that the function \( f(x) = x^x \) is increasing - except this is not true!
Instead, since \( \ln(x) \) is increasing, the above inequality is equivalent to showing that
\[
\sin(x)\ln(\sin(x)) < \cos(x)\ln(\cos(x)).
\]
Since \( \sin(x) > 0 \) on \((0, \frac{\pi}{4})\), this is equivalent to

\[
\ln(\cos x) - \tan(x) \ln(\sin x) > 0.
\]

The function \( \ln(x) \) is concave (check it!) and so

\[
\lambda \ln(a) + (1 - \lambda) \ln(b) < \ln(\lambda a + (1 - \lambda) b)
\]

for all \( a, b \in (0, \infty) \). Taking \( a = \sin(x) \) and \( b = \sin(x) + \cos(x) \) and \( \lambda = \tan(x) \), we get

\[
\tan(x) \ln(\sin(x)) + (1 - \tan(x)) \ln(\sin(x) + \cos(x)) < \ln(\tan(x) \sin(x) + (1 - \tan(x))(\sin(x) + \cos(x)))
\]

\[
= \ln(\cos(x)).
\]

We are done by noticing that \( (1 - \tan(x)) \ln(\sin(x) + \cos(x)) > 0 \) since \( \sin(x) + \cos(x) = \sqrt{2}\cos(\pi/4 - x) > 1 \).

4. Let \( k \) be a fixed positive integer. The \( n \)-th derivative of \( \frac{1}{x^k - 1} \) has the form \( \frac{P_n(x)}{(x^k - 1)^{n+1}} \) where \( P_n(x) \) is a polynomial. Find \( P_n(1) \).

**Solution:** By differentiating \( P_n(x)/(x^k - 1)^{n+1} \), we find that \( P_{n+1}(x) = (x^k - 1)P'_n(x) - (n + 1)kx^{k-1}P_n(x) \); substituting \( x = 1 \) yields \( P_{n+1}(1) = - (n + 1)kP_n(1) \). Since \( P_0(1) = 1 \), an easy induction gives \( P_n(1) = (-k)^n n! \) for all \( n \geq 0 \).

Note: one can also argue by expanding in Taylor series around 1. Namely, we have

\[
\frac{1}{x^k - 1} = \frac{1}{k(x - 1)} + \cdots = \frac{1}{k} (x - 1)^{-1} + \cdots,
\]

so

\[
\frac{d^n}{dx^n} \frac{1}{x^k - 1} = \frac{(-1)^n n!}{k(x - 1)^{-n-1}}
\]

and

\[
P_n(x) = (x^k - 1)^{n+1} \frac{d^n}{dx^n} \frac{1}{x^k - 1}
\]

\[
= (k(x - 1) + \cdots)^{n+1} \left( \frac{(-1)^n n!}{k} (x - 1)^{-n-1} + \cdots \right)
\]

\[
= (-k)^n n! + \cdots.
\]
5. Let $f : [a, b] \to \mathbb{R}$ be a convex function. Show that $f$ is continuous on $(a, b)$.

**Solution:** Fix $a < x_0 < b$. Let $\alpha$ and $\beta$ be two limit points of $f$ at $x_0$: $\alpha$ from the left and $\beta$ from the right. We want to prove that they are equal. If not, without loss of generality we can assume that $\alpha < \beta$. Choose $x < x_0$ and $y > x_0$ such that $|f(x) - \alpha|$ and $|f(y) - \beta|$ are small. Because $\beta$ is a limit point, there exist points on the graph of $f$ close to $(x_0, \beta)$ and hence above the line segment from $(x, f(x))$ to $(y, f(y))$. This contradicts the convexity of $f$. Hence $\alpha = \beta$. Hence $f$ has a limit at $x_0$. Redo the argument for $x = x_0$ to conclude that the limit is equal to $f(x_0)$. Hence $f$ is continuous at $x_0$. 