2023 - ISU Putnam Practice Set 3 - Solutions

Thursday, September 21, 2023

Continuity

1. Let $f : R \to R$ be a continuous function satisfying $f(x) = f(x^2)$ for all $x \in R$. Prove that f is constant.

   Solution: The condition from the statement implies that $f(x) = f(-x)$, so it suffices to check that f is constant on $[0,\infty)$. For $x \geq 0$, define the recursive sequence (x_n), by $x_0 = x$, and $x_{n+1} = \sqrt{x_n}$, for $n \geq 0$. Then

   $$f(x_0) = f(x_1) = f(x_2) = \cdots = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n).$$

   And $\lim_{n\to\infty} x_n = 1$ if $x > 0$. It follows that f is constant and the problem is solved.

2. Let a and b be real numbers in the interval $\left(0, \frac{1}{2}\right)$ and let f be a continuous real-valued function such that

   $$f(f(x)) = af(x) + bx, \text{ for all } x \in \mathbb{R}.$$

   Prove that $f(0) = 0$.

   Solution: From the given condition, it follows that f is one-to-one. Indeed, if $f(x) = f(y)$, then $f(f(x)) = f(f(y))$, so $bx = by$, which implies $x = y$. Because f is continuous and one-to-one, it is strictly monotonic. We will show that f has a fixed point. Assume by way of contradiction that this is not the case. So either $f(x) > x$ for all x, or $f(x) < x$ for all x. In the first case f must be strictly increasing, and then we have the chain of implications

   $$f(x) > x \Rightarrow f(f(x)) > f(x) \Rightarrow af(x) + bx > f(x) \Rightarrow f(x) < \frac{bx}{1-a},$$

   for all $x \in \mathbb{R}$. In particular, $f(1) < \frac{b}{1-a} < 1$, contradicting our assumption.

   In the second case the simultaneous inequalities $f(x) < x$ and $f(f(x)) < f(x)$ show that f must be strictly increasing again. Again we have a chain of implications

   $$f(x) < x \Rightarrow f(f(x)) < f(x) \Rightarrow f(x) > af(x) + bx \Rightarrow f(x) > \frac{bx}{1-a},$$

   for all $x \in \mathbb{R}$. In particular, $f(-1) > -\frac{b}{1-a} > -1$, again a contradiction.

In conclusion, there exists a real number c such that $f(c) = c$. The condition $f(f(c)) = af(c) + bc$ implies $c = ac + bc$; thus $c(a+b-1) = 0$. It follows that $c = 0$ (because $a+b < 1/2+1/2 = 1$), and we obtain $f(0) = 0$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous decreasing function. Prove that the system

$$x = f(y),$$
$$y = f(z),$$
$$z = f(x)$$

has a unique solution.

Solution: The fact that f is decreasing implies immediately that

$$lim_{x\to\infty}(f(x)-x) = -\infty \text{ and } lim_{x\to-\infty}(f(x)-x) = \infty.$$

By the intermediate value property, there is x_0 such that $f(x_0) = x_0$. The function cannot have another fixed point because if x and y are fixed points, with $x < y$, then $x = f(x) \geq f(y) = y$, impossible. The triple (x_0, x_0, x_0) is a solution to the system. And if (x, y, z) is a solution then $f(f(f(x))) = x$. The function $f \circ f \circ f$ is also continuous and decreasing, so it has a unique fixed point. And this fixed point can only be x_0. Therefore, $x = y = z = x_0$, proving that the solution is unique.

4. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(0) = 1$ and $f(2x) - f(x) = x$, for all $x \in \mathbb{R}$.

Solution: We have

$$\begin{aligned} f(x) - f\left(\frac{x}{2}\right) &= \frac{x}{2} \\ f\left(\frac{x}{2}\right) - f\left(\frac{x}{4}\right) &= \frac{x}{4} \\ f\left(\frac{x}{4}\right) - f\left(\frac{x}{8}\right) &= \frac{x}{8} \\ &\vdots \\ f\left(\frac{x}{2^{n-1}}\right) - f\left(\frac{x}{2^n}\right) &= \frac{x}{2^n}. \end{aligned}$$

Summing up we get

$$f(x) - f\left(\frac{x}{2^n}\right) = x\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}\right).$$

Taking the limit $n \to \infty$ and using the continuity of f, we get

$$f(x) - 1 = x,$$

and hence $f(x) = x + 1$ is the unique solution.

5. Let $f : [-1, 1] \to \mathbb{R}$ be a continuous function such that

   (i) $f(x) = \frac{2-x^2}{2} f\left(\frac{x^2}{2-x^2}\right)$ for every x in $[-1, 1]$,

   (ii) $f(0) = 1$, and

   (iii) $\lim_{x\to 1^-} \frac{f(x)}{\sqrt{1-x}}$ exists and is finite.

   Prove that f is unique, and express $f(x)$ in closed form.

   Solution: (Putnam 2012 A3) We will prove that $f(x) = \sqrt{1-x^2}$ for all $x \in [-1, 1]$. Define $g : (-1, 1) \to \mathbb{R}$ by $g(x) = f(x)/\sqrt{1-x^2}$. Plugging $f(x) = g(x)\sqrt{1-x^2}$ into equation (i) and simplifying yields

$$g(x) = g\left(\frac{x^2}{2-x^2}\right) \tag{1}$$

   for all $x \in (-1, 1)$. Now fix $x \in (-1, 1)$ and define a sequence $\{a_n\}_{n=1}^{\infty}$ by $a_1 = x$ and $a_{n+1} = \frac{a_n^2}{2-a_n^2}$. Then $a_n \in (-1, 1)$ and thus $|a_{n+1}| \le |a_n|^2$ for all n. It follows that $\{|a_n|\}$ is a decreasing sequence with $|a_n| \le |x|^n$ for all n, and so $\lim_{n\to\infty} a_n = 0$. Since $g(a_n) = g(x)$ for all n by (1) and g is continuous at 0, we conclude that $g(x) = g(0) = f(0) = 1$. This holds for all $x \in (-1, 1)$ and thus for $x = \pm 1$ as well by continuity. The result follows.