Sequences and Series

1. Find

$$\lim_{n \to \infty} |\sin \pi \sqrt{n^2 + n + 1}|.$$

**Solution:** Since $|\sin x|$ is periodic with period $\pi$,

$$\lim_{n \to \infty} |\sin \pi \sqrt{n^2 + n + 1}| = \lim_{n \to \infty} |\sin \pi \left(\sqrt{n^2 + n + 1} - n\right)|.$$

But

$$\lim_{n \to \infty} \left(\sqrt{n^2 + n + 1} - n\right) = \lim_{n \to \infty} \frac{n^2 + n + 1 - n}{\sqrt{n^2 + n + 1} + n} = \frac{1}{2}.$$

By continuity, the limit is $|\sin \frac{\pi}{2}| = 1$.

2. Let $S = \{x_1, x_2, \ldots, x_n, \ldots\}$ be the set of all positive integers that do not contain the digit 9 in their decimal representation. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{x_n}$$

converges.

**Solution:** There are exactly $8 \cdot 9^{n-1}$ $n$-digit numbers in $S$ (the first digit can be chosen in 8 ways, and all others in 9 ways). The least of these numbers is $10^n$. We can therefore write

$$\sum_{x_j<10^n} \frac{1}{x_j} = \sum_{i=1}^{n} \sum_{10^{i-1} \leq x_j < 10^i} \frac{1}{x_j}$$

$$< \sum_{i=1}^{n} \sum_{10^{i-1} \leq x_j < 10^i} \frac{1}{10^{i-1}}$$

$$= \sum_{i=1}^{n} \frac{8 \cdot 9^{i-1}}{10^{i-1}}$$

$$= 80 \left(1 - \left(\frac{9}{10}\right)^n\right).$$

Letting $n \to \infty$ we get that the sum is at most 80.
3. Is the number
\[ \sum_{n=1}^{\infty} \frac{1}{2n^2} \]

rational?

**Solution:** The series is convergent because it is bounded from above by the geometric series with ratio \( \frac{1}{2} \). Assume that its sum is a rational number \( \frac{a}{b} \). Choose \( n \) such that \( b < 2^n \). Then

\[
\frac{a}{b} - \sum_{k=1}^{n} \frac{1}{2k^2} = \sum_{k \geq n+1} \frac{1}{2k^2}.
\]

But the sum \( \sum_{k=1}^{n} \frac{1}{2k^2} \) is equal to \( \frac{m}{2^n} \) for some integer \( m \). Hence

\[
\frac{a}{b} - \sum_{k=1}^{n} \frac{1}{2k^2} = \frac{a - m}{b - 2n^2} > \frac{1}{2n^2 b} > \frac{1}{2n^2 + n} > \frac{1}{2(n+1)^2 - 1} = \sum_{k \geq (n+1)^2} \frac{1}{2k} > \sum_{k \geq n+1} \frac{1}{2k^2},
\]

which is a contradiction. Thus the sum is irrational.

4. Let
\[ a_n = \sqrt{1 + \left(1 + \frac{1}{n}\right)^2} + \sqrt{1 + \left(1 - \frac{1}{n}\right)^2}, \]

for \( n \geq 1 \). Prove that
\[ \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{20}} \]

is a positive integer.
Solution: We have

\[
\frac{1}{\sqrt{1 + \left(1 + \frac{1}{n}\right)^2} + \sqrt{1 + \left(1 - \frac{1}{n}\right)^2}} = \frac{\sqrt{1 + \left(1 + \frac{1}{n}\right)^2} - \sqrt{1 + \left(1 - \frac{1}{n}\right)^2}}{\frac{4}{n}}
\]

\[
= \frac{1}{4} \left( \sqrt{n^2 + (n+1)^2} - \sqrt{n^2 + (n-1)^2} \right)
\]

\[
= \frac{1}{4} (b_{n+1} - b_n),
\]

where \( b_n = \sqrt{n^2 + (n-1)^2} \). Hence the sum telescopes to \( \frac{1}{4}(29 - 1) = 7 \).

5. Let \( A \) be a positive real number. What are the possible values of \( \sum_{j=0}^{\infty} x_j^2 \), given that \( x_0, x_1, \ldots \) are positive numbers for which \( \sum_{j=0}^{\infty} x_j = A \)?

Solution Putnam 2000 A1: The possible values comprise the interval \((0, A^2)\).

To see that the values must lie in this interval, note that

\[
\left( \sum_{j=0}^{m} x_j \right)^2 = \sum_{j=0}^{m} x_j^2 + \sum_{0 \leq j < k \leq m} 2x_j x_k,
\]

so \( \sum_{j=0}^{m} x_j^2 \leq A^2 - 2x_0 x_1 \). Letting \( m \to \infty \), we have \( \sum_{j=0}^{\infty} x_j^2 \leq A^2 - 2x_0 x_1 < A^2 \).

To show that all values in \((0, A^2)\) can be obtained, we use geometric progressions with \( x_1/x_0 = x_2/x_1 = \cdots = d \) for variable \( d \). Then \( \sum_{j=0}^{\infty} x_j = x_0/(1-d) \) and

\[
\sum_{j=0}^{\infty} x_j^2 = \frac{x_0^2}{1-d^2} = \frac{1-d}{1+d} \left( \sum_{j=0}^{\infty} x_j \right)^2.
\]

As \( d \) increases from 0 to 1, \( (1-d)/(1+d) \) decreases from 1 to 0. Thus if we take geometric progressions with \( \sum_{j=0}^{\infty} x_j = A \), \( \sum_{j=0}^{\infty} x_j^2 \) ranges from 0 to \( A^2 \). Thus the possible values are indeed those in the interval \((0, A^2)\), as claimed.