2021 - ISU Putnam Practice Set 3 - Solutions

Friday, October 7, 2022

Sequences and Series

1. Find

$$\lim_{n\to\infty} |\sin \pi \sqrt{n^2+n+1}|.$$

Solution: Since $|\sin x|$ is periodic with period π,

$$\lim_{n\to\infty} |\sin \pi \sqrt{n^2+n+1}| = \lim_{n\to\infty} |\sin \pi \left(\sqrt{n^2+n+1}-n\right)|.$$

But

$$\lim_{n\to\infty} (\sqrt{n^2+n+1}-n) = \lim_{n\to\infty} \frac{n^2+n+1-n}{\sqrt{n^2+n+1}+n} = \frac{1}{2}.$$

By continuity, the limit is $|\sin \frac{\pi}{2}| = 1$.

2. Let $S = \{x_1, x_2, \ldots, x_n, \ldots\}$ be the set of all positive integers that do not contain the digit 9 in their decimal representation. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{x_n}$$

converges.

Solution: There are exactly $8 \cdot 9n-1$ n-digit numbers in S (the first digit can be chosen in 8 ways, and all others in 9 ways). The least of these numbers is $10n$. We can therefore write

$$\begin{aligned}
\sum_{x_j<10^n} \frac{1}{x_j} &= \sum_{i=1}^{n} \sum_{10^{i-1}\le x_j<10^i} \frac{1}{x_j} \\
&< \sum_{i=1}^{n} \sum_{10^{i-1}\le x_j<10^i} \frac{1}{10^{i-1}} \\
&= \sum_{i=1}^{n} \frac{8\cdot 9^{i-1}}{10^{i-1}} \\
&= 80\left(1-\left(\frac{9}{10}\right)^n\right).
\end{aligned}$$

Letting $n \to \infty$ we get that the sum is at most 80.

3. Is the number

$$\sum_{n=1}^{\infty} \frac{1}{2^{n^2}}$$

rational?

Solution: The series is convergent because it is bounded from above by the geometric series with ratio $\frac{1}{2}$. Assume that its sum is a rational number $\frac{a}{b}$. Choose n such that $b < 2^n$. Then

$$\frac{a}{b} - \sum_{k=1}^{n} \frac{1}{2^{k^2}} = \sum_{k \geq n+1} \frac{1}{2^{k^2}}.$$

But the sum $\sum_{k=1}^{n} \frac{1}{2^{k^2}}$ is equal to $\frac{m}{2^{n^2}}$ for some integer m. Hence

$$\begin{aligned} \frac{a}{b} - \sum_{k=1}^{n} \frac{1}{2^{k^2}} &= \frac{a}{b} - \frac{m}{2^{n^2}} \\ &> \frac{1}{2^{n^2} b} \\ &> \frac{1}{2^{n^2+n}} \\ &> \frac{1}{2^{(n+1)^2-1}} \\ &= \sum_{k \geq (n+1)^2} \frac{1}{2^k} \\ &> \sum_{k \geq n+1} \frac{1}{2^{k^2}}, \end{aligned}$$

which is a contradiction. Thus the sum is irrational.

4. Let

$$a_n = \sqrt{1 + \left(1 + \frac{1}{n}\right)^2} + \sqrt{1 + \left(1 - \frac{1}{n}\right)^2},$$

for $n \geq 1$. Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{20}}$$

is a positive integer.

Solution: We have

$$
\begin{aligned}
\frac{1}{\sqrt{1+\left(1+\frac{1}{n}\right)^2}+\sqrt{1+\left(1-\frac{1}{n}\right)^2}} &= \frac{\sqrt{1+\left(1+\frac{1}{n}\right)^2}-\sqrt{1+\left(1-\frac{1}{n}\right)^2}}{1+\left(1+\frac{1}{n}\right)^2-1-\left(1-\frac{1}{n}\right)^2} \\
&= \frac{\sqrt{1+\left(1+\frac{1}{n}\right)^2}-\sqrt{1+\left(1-\frac{1}{n}\right)^2}}{\frac{4}{n}} \\
&= \frac{1}{4}\left(\sqrt{n^2+(n+1)^2}-\sqrt{n^2+(n-1)^2}\right) \\
&= \frac{1}{4}(b_{n+1}-b_n),
\end{aligned}
$$

where $b_n = \sqrt{n^2+(n-1)^2}$. Hence the sum telescopes to $\frac{1}{4}(29-1) = 7$.

5. Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \ldots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

   Solution Putnam 2000 A1: The possible values comprise the interval $(0, A^2)$.

   To see that the values must lie in this interval, note that

$$
\left(\sum_{j=0}^{m} x_j\right)^2 = \sum_{j=0}^{m} x_j^2 + \sum_{0 \le j < k \le m} 2x_j x_k,
$$

   so $\sum_{j=0}^{m} x_j^2 \le A^2 - 2x_0x_1$. Letting $m \to \infty$, we have $\sum_{j=0}^{\infty} x_j^2 \le A^2 - 2x_0x_1 < A^2$.

   To show that all values in $(0, A^2)$ can be obtained, we use geometric progressions with $x_1/x_0 = x_2/x_1 = \cdots = d$ for variable d. Then $\sum_{j=0}^{\infty} x_j = x_0/(1-d)$ and

$$
\sum_{j=0}^{\infty} x_j^2 = \frac{x_0^2}{1-d^2} = \frac{1-d}{1+d}\left(\sum_{j=0}^{\infty} x_j\right)^2.
$$

   As d increases from 0 to 1, $(1-d)/(1+d)$ decreases from 1 to 0. Thus if we take geometric progressions with $\sum_{j=0}^{\infty} x_j = A$, $\sum_{j=0}^{\infty} x_j^2$ ranges from 0 to A^2. Thus the possible values are indeed those in the interval $(0, A^2)$, as claimed.