2021 - ISU Putnam Practice Set 3 - Solutions

Wednesday, September 22, 2021

Calculus 1

1. Let $f : [a,b] \to [a,b]$ be a continuous function. Show that f has a fixed point; i.e. show that there is a $c \in [a,b]$ with $f(c) = c$.

   Solution: Apply the IVT to $g(x) = f(x) - x$. Since $f(a) \geq a$, $g(a) \geq 0$; since $f(b) \leq b$, $g(b) \leq 0$. Thus there is a $c \in [a,b]$ with $f(c) = 0$, i.e. $f(c) = c$.

2. Find all positive real solutions to $2^x = x^2$.

   Solution: Taking logs, we get the equation $x\log(2) - 2\log(x) = 0$. Set $f(x) = x\log(2) - 2\log(x)$. Then $f'(x) = \log(2) - \frac{2}{x}$, which has a unique zero at $\frac{2}{\log(2)}$. For $x > \frac{2}{\log(2)}$, $f'(x) > 0$ and for $x < \frac{2}{\log(2)}$, $f'(x) < 0$. Since $\lim_{x\to 0} f(x) = \lim_{x\to\infty} f(x) = \infty$, either $f(x) = 0$ has two or zero real solutions. Since $f(2) = f(4) = 0$, there are exactly 2 real solutions.

3. Show that not all zeros of the polynomial $P(x) = x^4 - \sqrt{7}x^3 + 4x^2 - \sqrt{22}x + 15$ are real.

   Solution: If all 4 zeros of $P(x)$ are real, then by MVT (Rolle's Theorem), all three zeros of $P'(x)$ are real and both zeros of $P''(x) = 12x^2 - 6\sqrt{7}x + 8$ are real. By its discriminant is $6^2 \cdot 7 - 4 \cdot 12 \cdot 8 = -132$ is negative - a contradiction.

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying:

   $$|f(x) - f(y)| \leq |x - y|^2.$$

   for all $x, y \in \mathbb{R}$.

   Solution: The above inequality implies that

   $$\frac{|f(x) - f(y)|}{|x - y|} \leq |x - y|$$

   for $x \neq y$. The righthand side approaches 0 as $y \to x$, so the lefthand side does as well, but the limit of the lefthand side is $f'(x)$. So $f'(x) = 0$ for all x and thus f must be constant. Clearly all constant functions satisfy the inequality, so that is our answer.

5. Let f be a three times differentiable function (defined on $\mathbb{R}$ and real-valued) such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.

   Solution (B1 2015): Let $g(x) = e^{x/2}f(x)$. Then g has at least 5 distinct real zeroes, and by repeated applications of Rolle's theorem, g', g'', g''' have at least $4, 3, 2$ distinct real zeroes, respectively. But

$$g'''(x) = \frac{1}{8}e^{x/2}(f(x) + 6f'(x) + 12f''(x) + 8f'''(x))$$

   and $e^{x/2}$ is never zero, so we obtain the desired result.