2023 - ISU Putnam Practice Set 2 - Solutions

Thursday, September 14, 2023

Inequalities

1. Let $ABCD$ be a convex cyclic quadrilateral. Prove that

$$|AB-CD|+|AD-BC| \geq 2|AC-BD|.$$

Solution: The first idea is to simplify the problem and prove separately the inequalities $|AB-|CD| \geq |AC-BD|$ and $|AD-BC| \geq |AC-BD|$. Because of symmetry it suffices to prove the first. Let M be the intersection of the diagonals AC and BD. For simplicity, let $AM = x$, $BM = y$, $AB = z$. By the similarity of triangles MAB and MDC there exists a positive number k such that $DM = kx$, $CM = ky$, and $CD = kz$. Then $|AB-CD| = |k-1|z$ and $|AC-BD| = |(kx+y)-(ky+x)| = |k-1|\cdot|x-y|$. By the triangle inequality, $|x-y| \leq z$, which implies $|AB-CD| \geq |AC-BD|$, completing the proof.

2. Show that all real roots of the polynomial $P(x) = x^5 - 10x + 35$ are negative.

Solution: Because $P(x)$ has odd degree, it has a real zero r. If $r > 0$, then by the AM-GM inequality

$$\frac{r^5+1+1+1+2^5}{5} \geq \sqrt[5]{r^5\cdot 1\cdot 1\cdot 1\cdot 2^5} = 2r,$$

and hence

$$P(r) = r^5+1+1+1+2^5-5\cdot 2\cdot r \geq 0.$$

And the inequality is strict since $1 \neq 2$. Hence $r < 0$, as desired.

2nd Solution by Carson Givens: Observe that the nonzero coefficients have sign sequence: $+,-,+$. As there are two consecutive sign changes, Decarte's sign rule implies that there are either 0 or 2 real positive roots. This isn't really needed since we can use calculus instead.

Since $P'(x) = 5x^4 - 10$, $P(x)$ has local maximum at $-\sqrt[4]{2}$ with value $-2^{5/4}+10\cdot 2^{1/4}+35 > 0$ and a local minimum at $\sqrt[4]{2}$ with value $2^{5/4}-10\cdot 2^{1/4}+35 > 0$. It follows that any roots of $P(x)$ must be less than $-\sqrt[4]{2}$.

3. Let a,b,c be the side lengths of a triangle with perimeter 2. Prove that

$$1 < ab+bc+ca-abc \leq \frac{28}{27}.$$

Solution: The inequality from the statement is equivalent to that is,

$$0 < 1-(a+b+c)+ab+bc+ca-abc < \frac{1}{27},$$

that is

$$0 < (1-a)(1-b)(1-c) \leq \frac{1}{27}.$$

From the triangle inequalities $a+b>c$, $b+c>a$, $a+c>b$ and the condition $a+b+c=2$ it follows that $0<a,b,c<1$. The inequality on the left is now evident, and the one on the right follows from the AM-GM inequality

$$\sqrt[3]{xyz} \leq \frac{x+y+z}{3}$$

applied to $x=1-a$, $y=1-b$, $z=1-c$.

Second Solution inspired by Matthew Dorang: Proceed as above to get the equivalent inequalities:

$$0 < (1-a)(1-b)(1-c) \leq \frac{1}{27}.$$

By Heron's formula, the area A of this triangle is exactly $A=\sqrt{(1-a)(1-b)(1-c)}$. Since $A>0$, we get the left-hand inequality. It is also clear that A is maximized when our triangle is equilateral with $a=b=c=\frac{2}{3}$, in which case $A=\frac{\sqrt{3}}{9}$. Squaring both sides gives the right-hand inequality.

4. Let a,b,c be the side lengths of a triangle with the property that for any positive integer n, the numbers a^n , b^n , c^n can also be the side lengths of a triangle. Prove that the triangle is necessarily isosceles.

Solution: Let c be the largest side. By the triangle inequality, $c^n < a^n+b^n$ for all $n \geq 1$. This is equivalent to

$$1 \leq \left(\frac{a}{c}\right)^n + \left(\frac{b}{c}\right)^n, \qquad n \geq 1.$$

If $a<c$ and $b<c$, then by letting $n \to \infty$, we obtain $1<0$, impossible. Hence one of the other two sides equals c, and the triangle is isosceles.

5. Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

Solution: (Putnam 2004 B2) **First solution:** We have

$$(m+n)^{m+n} > \binom{m+n}{m} m^m n^n$$

because the binomial expansion of $(m+n)^{m+n}$ includes the term on the right as well as some others. Rearranging this inequality yields the claim.

Remark: One can also interpret this argument combinatorially. Suppose that we choose $m+n$ times (with replacement) uniformly randomly from a set of $m+n$ balls, of which m are red and n are blue. Then the probability of picking each ball exactly once is $(m+n)!/(m+n)^{m+n}$. On the other hand, if p is the probability of picking exactly m red balls, then $p < 1$ and the probability of picking each ball exactly once is $p(m^m/m!)(n^n/n!)$.

Second solution: (by David Savitt) Define

$$S_k = \{i/k : i = 1, \dots, k\}$$

and rewrite the desired inequality as

$$\prod_{x \in S_m} x \prod_{y \in S_n} y > \prod_{z \in S_{m+n}} z.$$

To prove this, it suffices to check that if we sort the multiplicands on both sides into increasing order, the i-th term on the left side is greater than or equal to the i-th term on the right side. (The equality is strict already for $i = 1$, so you do get a strict inequality above.)

Another way to say this is that for any i, the number of factors on the left side which are less than $i/(m+n)$ is less than i. But since $j/m < i/(m+n)$ is equivalent to $j < im/(m+n)$, that number is

$$\begin{aligned} &\left\lceil \frac{im}{m+n} \right\rceil - 1 + \left\lceil \frac{in}{m+n} \right\rceil - 1 \\ &\leq \frac{im}{m+n} + \frac{in}{m+n} - 1 = i - 1. \end{aligned}$$

Third solution: Put $f(x) = x(\log(x+1) - \log x)$; then for $x > 0$,

$$\begin{aligned} f'(x) &= \log(1+1/x) - \frac{1}{x+1} \\ f''(x) &= -\frac{1}{x(x+1)^2}. \end{aligned}$$

Hence $f''(x) < 0$ for all x; since $f'(x) \to 0$ as $x \to \infty$, we have $f'(x) > 0$ for $x > 0$, so f is strictly increasing.

Put $g(m) = m \log m - \log(m!)$; then $g(m+1) - g(m) = f(m)$, so $g(m+1) - g(m)$ increases with m. By induction, $g(m+n) - g(m)$ increases with n for any positive integer n, so in particular

$$\begin{aligned} g(m+n) - g(m) &> g(n) - g(1) + f(m) \\ &\geq g(n) \end{aligned}$$

since $g(1) = 0$. Exponentiating yields the desired inequality.

Fourth solution: (by W.G. Boskoff and Bogdan Suceavă) We prove the claim by induction on $m+n$. The base case is $m = n = 1$, in which case the desired inequality is obviously true: $2!/2^2 = 1/2 < 1 = (1!/1^1)(1!/1^1)$. To prove the induction step, suppose $m+n > 2$; we must then have $m > 1$ or $n > 1$ or both. Because the desired result is symmetric in m and n, we may as well assume $n > 1$. By the induction hypothesis, we have

$$\frac{(m+n-1)!}{(m+n-1)^{m+n-1}} < \frac{m!}{m^m} \frac{(n-1)!}{(n-1)^{n-1}}.$$

To obtain the desired inequality, it will suffice to check that

$$\frac{(m+n-1)^{m+n-1}}{(m+n-1)!} \frac{(m+n)!}{(m+n)^{m+n}} < \frac{(n-1)^{n-1}}{(n-1)!} \frac{n!}{(n)^n}$$

or in other words

$$\left(1 - \frac{1}{m+n}\right)^{m+n-1} < \left(1 - \frac{1}{n}\right)^{n-1}.$$

To show this, we check that the function $f(x) = (1 - 1/x)^{x-1}$ is strictly decreasing for $x > 1$; while this can be achieved using the weighted arithmetic-geometric mean inequality, we give a simple calculus proof instead. The derivative of $\log f(x)$ is $\log(1 - 1/x) + 1/x$, so it is enough to check that this is negative for $x > 1$. An equivalent statement is that $\log(1-x) + x < 0$ for $0 < x < 1$; this in turn holds because the function $g(x) = \log(1-x) + x$ tends to 0 as $x \to 0^+$ and has derivative $1 - \frac{1}{1-x} < 0$ for $0 < x < 1$.