Differential Equations

1. A not uncommon mistake is to believe that the product rule for derivatives says that \((fg)' = f'g\). If \(f(x) = e^{x^2}\), determine whether there exists an open interval \((a, b)\) and a nonzero function \(g\) defined on \((a, b)\) such that this wrong product rule is true for \(f\) and \(g\) on \((a, b)\).

Solution Putnam 1988 A2: The answer is yes! We have to prove that for \(f(x) = e^{x^2}\), the equation \(f'g + fg' = f'g'\) has nontrivial solutions on some interval \((a, b)\). Explicitly, this is the first-order linear equation in \(g\):

\[
(1 - 2x)e^{x^2}g' + 2xe^{x^2}g = 0.
\]

Separating the variables, we obtain

\[
g' = \frac{2x}{2x - 1} = 1 + \frac{1}{2x - 1},
\]

which yields by integration \(\ln g(x) = x + \frac{1}{2}\ln|2x - 1| + C\). We obtain the one-parameter family of solutions

\[
g(x) = ae^{x}\sqrt{|2x - 1|}, a \in \mathbb{R},
\]

on any interval that does not contain \(\frac{1}{2}\).

2. Let \(f\) and \(g\) be differentiable functions on the real line satisfying the equation

\[
(f^2 + g^2)f' + (fg)g' = 0.
\]

Prove that \(f\) is bounded.

Solution: The idea is to integrate the equation using an integrating factor. If instead we had the first-order differential equation \((x^2 + y^2)dx + xydy = 0\), then the standard method finds \(x\) as an integrating factor. So if we multiply our equation by \(f\) to transform it into

\[
(f^3 + fg^2)f' + f^2gg' = 0,
\]

then the new equation is equivalent to

\[
\left(\frac{1}{4}f^4 + \frac{1}{2}f^2g^2\right)' = 0.
\]
Therefore, \( f \) and \( g \) satisfy

\[ f^4 + 2f^2g^2 = C \]

for some real constant \( C \). In particular, \( f \) is bounded, as otherwise \( f^4 \leq f^4 + 2f^2g^2 = C \) would be arbitrarily large.

3. Find all twice-differentiable functions defined on the entire real axis that satisfy \( f'(x)f''(x) = 0 \) for all \( x \).

**Solution:** The function \( f'(x)f''(x) \) is the derivative of \( \frac{1}{2}(f'(x))^2 \). The equation is therefore equivalent to

\[ f'(x)^2 = \text{constant}. \]

And because \( f'(x) \) is continuous, \( f''(x) \) itself must be constant, which means that \( f(x) \) is linear. Clearly, all linear functions are solutions.

4. Solve the differential equation

\[ (x - 1)y'' + (4x - 5)y' + (4x - 6)y = xe^{-2x}. \]

**Solution:** The associated homogeneous equation can be written as

\[ x(y'' + 4y' + 4y) - (y'' + 5y' + 6y) = 0. \]

The equations \( y'' + 4y' + 4y = 0 \) and \( y'' + 5y' + 6y = 0 \) have common solution \( y(x) = e^{-2x} \).

This will therefore be a solution to the above equation as well. To find a solution to the inhomogeneous equation, we use the method of variation of the constant. Set \( y(x) = C(x)e^{-2x} \).

The equation becomes

\[ (x - 1)C'' - C' = x, \]

which as a first order equation has the solution

\[ C'(x) = \lambda (x - 1) + (x - 1) \ln |x - 1| - 1. \]

Integrating, we obtain

\[ C(x) = \frac{1}{2} (x - 1)^2 \ln |x - 1| + \left( \frac{\lambda}{2} - \frac{1}{4} \right) (x - 1)^2 - x + C_1. \]

If we set \( C_2 = \frac{\lambda}{2} - \frac{1}{4} \), then the general solution to the original equation is

\[ y(x) = e^{-2x} \left[ C_1 + C_2(x - 1)^2 + \frac{1}{2} (x - 1)^2 \ln |x - 1| - x \right]. \]
5. Let $f$ be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where $g(x) \geq 0$ for all real $x$. Prove that $|f(x)|$ is bounded.

**Solution Putnam 1997 B2:** It suffices to show that $|f(x)|$ is bounded for $x \geq 0$, since $f(-x)$ satisfies the same equation as $f(x)$. But then

$$\frac{d}{dx} \left( (f(x))^2 + (f'(x))^2 \right) = 2f'(x)(f(x) + f''(x))$$

$$= -2xg(x)(f'(x))^2 \leq 0,$$

so that $(f(x))^2 \leq (f(0))^2 + (f'(0))^2$ for $x \geq 0$. 