2021 - ISU Putnam Practice Set 2 - Solutions

Friday, September 30, 2022

Differential Equations

1. A not uncommon mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine whether there exists an open interval (a,b) and a nonzero function g defined on (a,b) such that this wrong product rule is true for f and g on (a,b).

   Solution Putnam 1988 A2: The answer is yes! We have to prove that for $f(x) = e^{x^2}$, the equation $f'g + fg' = f'g'$ has nontrivial solutions on some interval (a,b). Explicitly, this is the first-order linear equation in g:

   $$(1-2x)e^{x^2}g' + 2xe^{x^2}g = 0.$$

   Separating the variables, we obtain

   $$\frac{g'}{g} = \frac{2x}{2x-1} = 1 + \frac{1}{2x-1},$$

   which yields by integration $\ln g(x) = x + \frac{1}{2}\ln|2x-1| + C$. We obtain the one-parameter family of solutions

   $$g(x) = ae^x\sqrt{|2x-1|}, a \in \mathbb{R},$$

   on any interval that does not contain $\frac{1}{2}$.

2. Let f and g be differentiable functions on the real line satisfying the equation

   $$(f^2+g^2)f' + (fg)g' = 0.$$

   Prove that f is bounded.

   Solution: The idea is to integrate the equation using an integrating factor. If instead we had the first-order differential equation $(x^2+y^2)dx + xydy = 0$, then the standard method finds x as an integrating factor. So if we multiply our equation by f to transform it into

   $$(f^3+fg^2)f' + f^2gg' = 0,$$

   then the new equation is equivalent to

   $$\left(\frac{1}{4}f^4 + \frac{1}{2}f^2g^2\right)' = 0.$$

Therefore, f and g satisfy

$$f^4 + 2f^2g^2 = C$$

for some real constant C. In particular, f is bounded, as otherwise $f^4 \leq f^4 + 2f^2g^2 = C$ would be arbitrarily large.

3. Find all twice-differentiable functions defined on the entire real axis that satisfy $f'(x)f''(x) = 0$ for all x.

   Solution: The function $f'(x)f''(x)$ is the derivative of $\frac{1}{2}(f'(x))^2$. The equation is therefore equivalent to

   $$f'(x)^2 = \text{constant}.$$

   And because $f'(x)$ is continuous, $f'(x)$ itself must be constant, which means that $f(x)$ is linear. Clearly, all linear functions are solutions.

4. Solve the differential equation

   $$(x-1)y'' + (4x-5)y' + (4x-6)y = xe^{-2x}.$$

   Solution: The associated homogeneous equation can be written as

   $$x(y'' + 4y' + 4y) - (y'' + 5y' + 6y) = 0.$$

   The equations $y'' + 4y' + 4y = 0$ and $y'' + 5y' + 6y = 0$ have common solution $y(x) = e^{-2x}$. This will therefore be a solution to the above equation as well. To find a solution to the inhomogeneous equation, we use the method of variation of the constant. Set $y(x) = C(x)e^{-2x}$. The equation becomes

   $$(x-1)C'' - C' = x,$$

   which as a first order equation has the solution

   $$C'(x) = \lambda(x-1) + (x-1)\ln|x-1| - 1.$$

   Integrating, we obtain

   $$C(x) = \frac{1}{2}(x-1)^2 \ln|x-1| + \left(\frac{\lambda}{2} - \frac{1}{4}\right)(x-1)^2 - x + C_1.$$

   If we set $C_2 = \frac{\lambda}{2} - \frac{1}{4}$, then the general solution to the original equation is

   $$y(x) = e^{-2x}\left[C_1 + C_2(x-1)^2 + \frac{1}{2}(x-1)^2 \ln|x-1| - x\right].$$

5. Let f be a twice-differentiable real-valued function satisfying

$$f(x)+f''(x)=-xg(x)f'(x),$$

where $g(x)\geq 0$ for all real x. Prove that $|f(x)|$ is bounded.

Solution Putnam 1997 B2: It suffices to show that $|f(x)|$ is bounded for $x\geq 0$, since $f(-x)$ satisfies the same equation as $f(x)$. But then

$$\begin{aligned}\frac{d}{dx}\left((f(x))^2+(f'(x))^2\right)&=2f'(x)(f(x)+f''(x))\\&=-2xg(x)(f'(x))^2\leq 0,\end{aligned}$$

so that $(f(x))^2\leq (f(0))^2+(f'(0))^2$ for $x\geq 0$.