2021 - ISU Putnam Practice Set 2 - Solutions

Wednesday, September 15, 2021

Determinants and Linear Algebra

1. Show that

$$\begin{pmatrix} (x^2+1)^2 & (xy+1)^2 & (xz+1)^2 \\ (xy+1)^2 & (y^2+1)^2 & (yz+1)^2 \\ (xz+1)^2 & (yz+1)^2 & (z^2+1) \end{pmatrix} = 2(x-y)^2(x-z)^2(y-z)^2.$$

Solution: Modulo a factor of 2, this matrix is the product of 2 Vandermonde matrices.

$$\det\begin{pmatrix} (x^2+1)^2 & (xy+1)^2 & (xz+1)^2 \\ (xy+1)^2 & (y^2+1)^2 & (yz+1)^2 \\ (xz+1)^2 & (yz+1)^2 & (z^2+1) \end{pmatrix} = \begin{pmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ x^2 & y^2 & z^2 \end{pmatrix}.$$

By the multiplicative property of determinants (and the fact that multiplying 1 row by 2 multiplies the determinant by a factor of 2), yields the above equation.

2. Find all numbers in the interval $[-2015, 2015]$ that can be equal to the determinant of an 11×11 matrix with entries equal to 1 or -1.

Solution: Let us consider a matrix with entries equal to ± 1. Its determinant is clearly an integer. Adding the first row to the second, third, ..., eleventh we transform the elements of these rows in either 0 or ± 2. The entries of these new rows are therefore divisible by 2, and factoring these out we deduce that the determinant of the matrix is a multiple of $2^{10} = 1024$. There are only three integers that are multiples of 2^{10} in the specified interval, namely $0, \pm 2^{10}$. Let us show that each can be the determinant of such a matrix. To obtain 0, just make two rows equal. To obtain 2^{10} take the matrix that has 1 on and above the main diagonal, and -1 elsewhere. To obtain -2^{10} take the negative of this matrix.

3. Prove that for any integers $x_1, x_2, \ldots, x_n$ and positive integers $k_1, k_2, \ldots, k_n$, the determant

$$\det\begin{pmatrix} x_1^{k_1} & x_2^{k_1} & \cdots & x_n^{k_1} \\ x_1^{k_2} & x_2^{k_2} & \cdots & x_n^{k_2} \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{k_n} & x_2^{k_n} & \cdots & x_n^{k_n} \end{pmatrix}$$

is divisible by $n!$.

Solution: View the determinant as a polynomial in the independent variables $x_1, x_2, \ldots, x_n$. Because whenever $x_i = x_j$ the determinant vanishes, it follows that the determinant is divisible by $x_i - x_j$, and therefore by the product

$$\prod_{1 \le i < j \le n}^{n} (x_i - x_j).$$

Because the k_is are positive, the determinant is also divisible by $x_1 x_2 \cdots x_n$. To solve the problem, it suffices to show that for any positive integers $x_1, x_2, \ldots, x_n$, the product

$$x_1 x_2 \cdots x_n \prod_{1 \le i < j \le n} (x_i - x_j)$$

is divisible by n!. This can be proved by induction on n.

A parity check proves the case $n = 2$. Assume that the property is true for any $n - 1$ integers and let us prove it for n. Either one of the numbers $x_1, x_2, \ldots, x_n$ is divisible by n, or, by the pigeonhole principle, the difference of two of them is divisible by n. In the first case we may assume that x_n is divisible by n, in the latter that $x_n - x_1$ is divisible by n. In either case,

$$x_1 x_2 \cdots x_{n-1} \prod_{1 \le i < j \le n-1} (x_i - x_j)$$

is divisible by $(n?1)!$, by the induction hypothesis. It follows that the whole product is divisible by $n \times (n?1)! = n!$ as desired. We are done.

4. Let M be an $n \times n$ complex matrix. Prove that there exist Hermitian matrices A and B such that $M = A + iB$.(A matrix X is called Hermitian if $\overline{X^t} = X$).

   Solution: If $M = A + iB$,then $\overline{M^t} = \overline{A^t} + i\overline{B^t} = A - iB$. So we should take

   $$A = \frac{1}{2}(M + \overline{M^t}) \text{ and } B = \frac{1}{2i}(M - \overline{M^t}),$$

   which are of course both Hermitian.

5. Let A be the $n \times n$ matrix whose entry in the ith row and the jth column is

   $$\frac{1}{\min(i, j)\}}$$

   for $1 \le i, j \le n$. Compute $\det(A)$.

Solution (2014 A2): Let $v_1,\dots,v_n$ denote the rows of A. The determinant is unchanged if we replace v_n by $v_n - v_{n-1}$, and then v_{n-1} by $v_{n-1}-v_{n-2}$, and so forth, eventually replacing v_k by $v_k - v_{k-1}$ for $k \geq 2$. Since v_{k-1} and v_k agree in their first $k-1$ entries, and the k-th entry of $v_k - v_{k-1}$ is $\frac{1}{k} - \frac{1}{k-1}$, the result of these row operations is an upper triangular matrix with diagonal entries $1, \frac{1}{2}-1, \frac{1}{3}-\frac{1}{2}, \dots, \frac{1}{n}-\frac{1}{n-1}$. The determinant is then

$$\begin{aligned}\prod_{k=2}^{n}\left(\frac{1}{k}-\frac{1}{k-1}\right) &= \prod_{k=2}^{n}\left(\frac{-1}{k(k-1)}\right)\\ &= \frac{(-1)^{n-1}}{(n-1)!n!}.\end{aligned}$$

Note that a similar calculation can be made whenever A has the form $A_{ij} = \min\{a_i, a_j\}$ for any monotone sequence $a_1,\dots,a_n$. Note also that the standard Gaussian elimination algorithm leads to the same upper triangular matrix, but the nonstandard order of operations used here makes the computations somewhat easier.

Remark: The inverse of A can be identified explicitly: for $n \geq 2$, it is the matrix B given by

$$B_{ij} = \begin{cases} -1 & i=j=1 \\ -2i^2 & 1<i=j<n \\ -(n-1)n & i=j=n \\ ij & |i-j|=1 \\ 0 & \text{otherwise.} \end{cases}$$

For example, for $n=5$,

$$B = \begin{pmatrix} -1 & 2 & 0 & 0 & 0 \\ 2 & -8 & 6 & 0 & 0 \\ 0 & 6 & -18 & 12 & 0 \\ 0 & 0 & 12 & -32 & 20 \\ 0 & 0 & 0 & 20 & -20 \end{pmatrix}.$$

Let C denote the matrix obtained from B by replacing the bottom-right entry with $-2n^2$ (for consistency with the rest of the diagonal). Expanding in minors along the bottom row produces a second-order recursion for $\det(C)$ solving to $\det(C) = (-1)^n (n!)^2$; a similar expansion then yields $\det(B) = (-1)^{n-1} n!(n-1)!$.

Remark: This problem and solution are due to one of us (Kedlaya). The statement appears in the comments on sequence A010790 (i.e., the sequence $(n-1)!n!$) in the On-Line Encyclopedia of Integer Sequences (`http://oeis.org`), attributed to Benoit Cloitre in 2002.