2023 - ISU Putnam Practice Set 1 - Solutions

Thursday, September 7, 2023

Polynomials 1

1. Find the zeros of the polynomial

$$P(x) = x^4 - 6x^3 + 18x^2 - 30x + 25$$

given that the sum of two of them is 4.

Solution: Denote the zeros of $P(x)$ by x_1, x_2, x_3, x_4, such that $x_1 + x_2 = 4$. The first Viete relation gives $x_1 + x_2 + x_3 + x_4 = 6$ and hence $x_3 + x_4 = 2$. The second relation gives

$$x_1x_2 + x_3x_4 + (x_1 + x_2)(x_3 + x_4) = 18,$$

and so $x_1x_2 + x_3x_4 = 10$. Combined with $x_1x_2x_3x_4 = 25$ shows that the products x_1x_2 and x_3x_4 are roots of the quadratic equation $u^2 - 10u + 25 = 0$. Thus $x_1x_2 = x_3x_4 = 5$. Therefore x_1, x_2 satisfy the quadratic equation $x^2 - 4x + 5 = 0$, while x_3, x_4 satisfy the quadratic equation $x^2 - 2x + 5 = 0$. We conclude that the zeros are $2 + i, 2 - i, 1 + 2i, 1 - 2i$.

2. Solve the system of equations

$$x + y + z = 1$$
$$xyz = 1,$$

given that x, y, z are complex numbers of absolute value equal to 1.

Solution: Taking conjugates of the first equation we get

$$\bar{x} + \bar{y} + \bar{z} = 1$$

and hence

$$1/x + 1/y + 1/z = 1.$$

Multiplying by xyz we get

$$xy + yz + xz = xyz = 1.$$

Therefore x, y, z are roots of the polynomial equation $t^3 - t^2 + t - 1 = 0$, which are $1, i, -i$. Any permutation of these three numbers is a solution to the original system of equations.

3. Find all polynomials whose coefficients are equal to either 1 or -1 and whose zeros are all real.

Solution: Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$. Denote its zeros by $x_1, x_2, \ldots, x_n$. The first two Viete's relations give

$$x_1 + x_2 + \cdots + x_n = -\frac{a_{n-1}}{a_n}$$

and

$$x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n = \frac{a_{n-2}}{a_n}.$$

Combining them we have

$$x_1^2 + x_2^2 + \cdots + x_n^2 = \left(\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right).$$

The only possibility is $x_1^2 + x_2^2 + \cdots + x_n^2 = 3$. Given that $x_1^2 x_2^2 \cdots x_n^2 = 1$, the AM-GM inequality yields $3 = x_1^2 + \cdots + x_n^2 \geq n\sqrt[n]{x_1^2 \cdots x_n^2} = n$. Therefore $n \geq 3$. Eliminating case by case, we find among linear polynomials $x+1$ and $x-1$ and among quadratic polynomials $x^2 + x - 1$ and $x^2 - x - 1$. When $n = 3$, we have equality above so all roots have the same absolute values. The polynomial should share a zero with its derivative. This is the case only for $x^3 + x^2 - x - 1$ and $x^3 - x^2 - x + 1$. Together with their negatives, these are all the desired polynomials.

4. The zeros of the polynomial $P(x) = x^3 - 10x + 11$ are u, v, and w. Determine the value of $\arctan(u) + \arctan(v) + \arctan(w)$.

Solution: Let $a = \arctan(u)$, $b = \arctan(v)$, and $c = \arctan(w)$. We are required to find $a + b + c$. The addition formula for the tangent of three angels is

$$\tan(a+b+c) = \frac{\tan(a) + \tan(b) + \tan(c) - \tan(a)\tan(b)\tan(c)}{1 - \tan(a)\tan(b) + \tan(b)\tan(c) + \tan(a)\tan(c))}.$$

This implies that

$$\tan(a+b+c) = \frac{u + u + w - uvw}{1 - (uv + vw + uw)}.$$

Using Viete's relations we get

$$u + v + w = 0$$

$$uv + vw + uw = -10$$

$$uvw = -11.$$

Therefore $\tan(a+b+c) = \frac{11}{1+10} = 1$. Thus $a+b+c = \frac{\pi}{4} + k\pi$ for some integer k.

Now since $uvw = -11 < 0$, either one or three of u, v, w are negative. Since $u+v+w = 0$, exactly two of u, v are positive and one is negative. Thus two of a, b, c are positive and one is negative. So one of a, b, c lies in the interval $(\frac{-pi}{2}, 0)$ and two lie in the interval $(0, \frac{\pi}{2})$. Hence k must be 0 and $a+b+c = \frac{\pi}{4}$.

5. Find all polynomials $P(x)$ with integer coefficients satisfying $P(P'(x)) = P'(P(x))$.

Solution: First suppose that $n \geq 2$. Write $P(x) = a_n x^n + \cdots + a_0$, $a_n \neq 0$. Then $P'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$. Identifying the coefficients of $x^{n(n-1)}$ in $P(P'(x)) = P'(P(x))$, we get

$$a_n^{n+1} n^n = a_n^n n.$$

This implies that $a_n n^{n-1} = 1$ and so $a_n = \frac{1}{n^{n-1}}$. Since a_n is an integer, n must be equal to 1, a contradiction.

If $n = 1$, say $P = ax + b$, then we should have $a^2 + b = a$ and hence $b = a - a^2$. Thus the answer is all polynomials of the form $ax + a - a^2$.