2021 - ISU Putnam Practice Set 1 - Solutions

Friday, September 23, 2022

Induction

1. Prove that $|\sin(nx)| \leq n|\sin(x)|$ for any real number x and any positive integer n.

   Solution: The base case is trivial. For the inductive step, use the sum formula for sin to write

$$\begin{aligned}
|\sin((k+1)x)| &= |\sin(kx+x)| \\
&= |\sin(kx)\cos(x)+\sin(x)\cos(kx)| \\
&\leq |\sin(kx)||\cos(x)|+|\sin(x)||\cos(kx)| \\
&\leq k|\sin(x)|+|\sin(x)| \\
&= (k+1)\sin(x).
\end{aligned}$$

   The first inequality is the triangle inequality; the second is the inductive hypothesis.

2. Prove that the Fibonacci sequence satisfies the identity

$$F_{2n+1} = F_{n+1}^2 + F_n^2, \text{ for } n \geq 0.$$

   Solution: We prove the more general identity:

$$F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n.$$

   Induct on n. The base cases are $n = 0$, in which case $F_{m+1} = F_{m+1} + 0$ holds trivially, and $n = 1$, in which case $F_{m+2} = F_{m+1} + F_m$ is the defining recursion for the Fibonacci sequence. Now assume $k \geq 2$ and that the relation above holds for $n < k$. Then

$$\begin{aligned}
F_{m+k+1} &= F_{m+k-1} + F_{m+k} \\
&= F_{m+1}F_k + F_mF_{k-1} + F_{m+1}F_{k-1} + F_mF_{k-2} \\
&= F_{m+1}(F_k + F_{k-1}) + F_m(F_{k-1} + F_{k-2}) \\
&= F_{m+1}F_{k+1} + F_mF_k,
\end{aligned}$$

   which completes the proof.

3. Show that any positive integer can be represented as $\pm 1^2 \pm 2^2 \pm \cdots \pm n^2$ for some positive integer n and some choice of signs.

Solution: Induct on the number to be represented. First note that

$$\begin{aligned} 1 &= +1^2 \\ 2 &= -1^2 - 2^2 - 3^2 + 4^2 \\ 3 &= -1^2 + 2^2 \\ 4 &= -1^2 - 2^2 + 3^2. \end{aligned}$$

The inductive step is "$P(n)$ implies $P(n+4)$" by way of the identity

$$m^2 - (m+1)^2 - (m+2)^2 + (m+3)^2 = 4.$$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$f\left(\frac{x_1 + x_2}{2}\right) = \frac{f(x_1) + f(x_2)}{2}$$

for any x_1, x_2. Prove that

$$f\left(\frac{x_1 + x_2 + \cdots x_n}{n}\right) = \frac{f(x_1) + f(x_2) + \cdots f(x_n)}{n}$$

for any $x_1, x_2, \ldots, x_n$.

Solution: By induction on k, one immediately gets the statement for $n = 2^k$:

$$\begin{aligned} f\left(\frac{x_1 + x_2 + \cdots + x_{2^{k+1}}}{2^{k+1}}\right) &= \frac{f\left(\frac{x_1+x_2+\cdots+x_{2^k}}{2^k}\right) + f\left(\frac{x_{2^k+1}+x_{2^k+2}+\cdots+x_{2^{k+1}}}{2^k}\right)}{2} \\ &= \frac{\frac{f(x_1)+f(x_2)+\cdots+f(x_{2^k})}{2^k} + \frac{f(x_{2^k+1})+f(x_{2^k+2})+\cdots+f(x_{2^{k+1}})}{2^k}}{2} \\ &= \frac{f(x_1) + f(x_2) + \cdots f(x_{2^{k+1}})}{2^{k+1}}. \end{aligned}$$

Now we work backwards. Assume the identity holds for n. Consider $x_1, \ldots, x_{n-1}$ and set $x_n = \frac{x_1+x_2+\cdots+x_{n-1}}{n-1}$. By assumption

$$\begin{aligned} f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) &= f\left(\frac{x_1 + x_2 + \cdots + x_{n-1} + \frac{x_1+x_2+\cdots+x_{n-1}}{n-1}}{n}\right) \\ &= \frac{f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f\left(\frac{x_1+x_2+\cdots+x_{n-1}}{n-1}\right)}{n}. \end{aligned}$$

This is the same as

$$f\left(\frac{x_1+x_2+\cdots+x_{n-1}}{n-1}\right)=\frac{1}{n}\left(f(x_1)+\cdots+f(x_{n-1})\right)+\frac{1}{n}f\left(\frac{x_1+x_2+\cdots+x_{n-1}}{n-1}\right).$$

Solving for the term on the left yields the desired equation.

5. Prove that $f(n)=1-n$ is the only integer-valued function defined on the integers that satisfies the following conditions.

   (i) $f(f(n))=n$, for all integers n;

   (ii) $f(f(n+2)+2)=n$ for all integers n;

   (iii) $f(0)=1$.

   Solution 1992 A1: If $f(n)=1-n$, then $f(f(n))=1-(1-n)=n$, so (i) holds. Similarly $f(f(n+2)+2)=1-[[1-(n+2)]+2]=n$, so (ii) holds. As $f(0)=1-0=1$, (iii) holds as well.

   Now suppose that f satisfies $(i)-(iii)$. By (i), $f(0)=1$ and $f(1)=f(f(0))=0$. By (ii), we have $f(f(f(n+2)+2))=f(n)$ while by (i) we have $f(f(f(n+2)+2))=f(n+2)+2$ and hence $f(n)-f(n+2)=2$. Induction (up and down $\mathbb{Z}$) forces $f(n)=1-n$.