2021 - ISU Putnam Practice Set 1 - Solutions

Wednesday, September 8, 2021

Games of No Chance

1. Choose a positive integer n. At each turn one of the players writes a positive integer that does not exceed n, the rule being that the player cannot write a divisor of a number already written. The player who cannot continue loses. Show that player one has a winning strategy.

   Solution: Assume to the contrary that player two has a winning strategy for any number player one writes. Then player one simply writes 1 first, the roles swap, and player one now has the winning strategy - a contradiction.

2. In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

   Solution (2002 A4): Denote the 3×3 matrix entries as:

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

   Then the determinant of M is

$$\det(M) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

   Rearrange the 3×3 grid with entries as follows:

a_{11}	a_{22}	a_{33}
a_{23}	a_{31}	a_{12}
a_{32}	a_{13}	a_{21}

   Then every term in $\det(M)$ shows up as a row or column above. As normal tic tac toe has an optimal strategy by player 2 to force a draw, player 0 can fill the tic tac toe grid in such a way that there is a 0 in each row and each column. The corresponding moves in the original grid force $\deg(M) = 0$ and thus player 0 has a winning strategy.

3. In (a version of) the game of Nim, two players start with a pile of n stones. On each turn, a player removes 1, 2, or 3 stones from the pile. The player to take the last stone wins. For which n does player 1 have a winning strategy?

   Solution: Player 1 has a winning strategy if n is not a multiple of 4. We prove this by induction on n. If $n = 1, 2$ or 3, then player 1 takes all the stones and wins. If $n = 4$, then no matter what player 1 does, player 2 has a winning strategy. Now suppose $n \geq 5$. If n is not a multiple of 4, then player one may remove $1, 2$ or 3 stones and leave player 2 with a multiple of 4 number of stones. Player 1 then has a winning strategy by induction. If n is divisible by 4, then no matter what player 1 does, player 2 can leave player 1 with a pile of n stones with n divisible by 4, for which player 2 has a winning strategy by induction.

4. Let k and n be integers with $1 \leq k < n$. Alice and Bob play a game with k pegs in a line of n holes. At the beginning of the game, the pegs occupy the k leftmost holes. A legal move consists of moving a single peg to any vacant hole that is further to the right. The players alternate moves, with Alice playing first. The game ends when the pegs are in the k rightmost holes, so whoever is next to play cannot move, and therefore loses. For what values of n and k does Alice have a winning strategy?

   Solution 2020 B2: Number the holes, from left to right, $1, 2, \ldots, n$. We first show that when k and n are both even, Bob has a winning strategy. In this case we can divide the holes into disjoint adjacent pairs $P_i = \{2i - 1, 2i\}$ with $1 \leq i \leq n/2$. At the beginning of the game the pegs completely occupy the holes in the leftmost k/2 pairs, and all the holes in the remaining pairs are vacant. Thus Alice?s first move must take a peg from an occupied pair of holes and place it in one of a vacant pair of holes. A winning strategy for Bob is to always take the other peg of the pair that Alice moved from and place it in the remaining hole of the pair that Alice moved to. Thus after each of Bob?s moves, each of the pairs P_i either has pegs in both holes or in neither, whereas after each of Alice?s moves, there are two of the pairs P_i with one peg each. In particular, Alice can never reach the ending position, and the game will end after one of Bob?s moves. If k and n are not both even, Alice always has a first move available which will leave Bob either with no moves at all, or with a position equivalent to the starting position of our game with even integers k_1 and n_1, $1 \leq k_1 < n_1$. Thus by the case discussed in the previous paragraph, Alice (as the second player from that position) has a winning strategy. Specifically, if k and n are both odd, Alice can move the peg in hole k to hole n, leaving $k_1 = k-1$ pegs at the beginning of a line of $n_1 = n-1$ remaining holes. (If $k = 1$, the game is then over.) If k is odd and n is even, Alice can move the peg in hole 1 to hole n, winning the game immediately if $k = 1$ and otherwise leaving $k_1 = k-1$ pegs at the beginning of a line of $n_1 = n-2$ remaining holes. Finally, if k is even and n is odd, Alice can move the peg in hole 1 to hole $k+1$, winning the game immediately if $n = k+1$ and otherwise leaving $k_1 = k$ pegs at the beginning of a line of $n_1 = n-1$ remaining holes. In each of these three cases, after making the indicated first move, Alice can use Bob?s strategy from the previous paragraph to win.

5. An integer n, unknown to you, has been randomly chosen in the interval $[1,2002]$ with uniform probability. Your objective is to select n in an **odd** number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you **must** guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than $2/3$.

   Solution 2002 B4: Use the following strategy: guess $1,3,4,6,7,9,\ldots$ until the target number n is revealed to be equal to or lower than one of these guesses. If $n \equiv 1 \pmod{3}$, it will be guessed on an odd turn. If $n \equiv 0 \pmod{3}$, it will be guessed on an even turn. If $n \equiv 2 \pmod{3}$, then $n+1$ will be guessed on an even turn, forcing a guess of n on the next turn. Thus the probability of success with this strategy is $1335/2002 > 2/3$.

   Note: for any positive integer m, this strategy wins when the number is being guessed from $[1,m]$ with probability $\frac{1}{m}\lfloor\frac{2m+1}{3}\rfloor$. We can prove that this is best possible as follows. Let a_m denote m times the probability of winning when playing optimally. Also, let b_m denote m times the corresponding probability of winning if the objective is to select the number in an even number of guesses instead. (For definiteness, extend the definitions to incorporate $a_0 = 0$ and $b_0 = 0$.)

   We first claim that $a_m = 1 + \max_{1\le k\le m}\{b_{k-1} + b_{m-k}\}$ and $b_m = \max_{1\le k\le m}\{a_{k-1} + a_{m-k}\}$ for $m \geq 1$. To establish the first recursive identity, suppose that our first guess is some integer k. We automatically win if $n = k$, with probability $1/m$. If $n < k$, with probability $(k-1)/m$, then we wish to guess an integer in $[1,k-1]$ in an even number of guesses; the probability of success when playing optimally is $b_{k-1}/(k-1)$, by assumption. Similarly, if $n < k$, with probability $(m-k)/m$, then the subsequent probability of winning is $b_{m-k}/(m-k)$. In sum, the overall probability of winning if k is our first guess is $(1 + b_{k-1} + b_{m-k})/m$. For optimal strategy, we choose k such that this quantity is maximized. (Note that this argument still holds if $k = 1$ or $k = m$, by our definitions of a_0 and b_0.) The first recursion follows, and the second recursion is established similarly.

We now prove by induction that $a_m = \lfloor (2m+1)/3 \rfloor$ and $b_m = \lfloor 2m/3 \rfloor$ for $m \geq 0$. The inductive step relies on the inequality $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x+y \rfloor$, with equality when one of x, y is an integer. Now suppose that $a_i = \lfloor (2i+1)/3 \rfloor$ and $b_i = \lfloor 2i/3 \rfloor$ for $i < m$. Then

$$\begin{aligned} 1 + b_{k-1} + b_{m-k} &= 1 + \left\lfloor \frac{2(k-1)}{3} \right\rfloor + \left\lfloor \frac{2(m-k)}{3} \right\rfloor \\ &\leq \left\lfloor \frac{2m}{3} \right\rfloor \end{aligned}$$

and similarly $a_{k-1} + a_{m-k} \leq \lfloor (2m+1)/3 \rfloor$, with equality in both cases attained, e.g., when $k = 1$. The inductive formula for a_m and b_m follows.