A FINITE CLASSIFICATION OF (x, y)-PRIMARY IDEALS OF LOW MULTIPLICITY

PAOLO MANTERO AND JASON MCCULLOUGH

ABSTRACT. Let S be a polynomial ring over an algebraically closed field k. Let x and y denote linearly independent linear forms in S so that $\mathfrak{p} = (x, y)$ is a height two prime ideal. This paper concerns the structure of $\mathfrak{p}$-primary ideals in S. Huneke, Seceleanu, and the authors showed that for $e \geq 3$, there are infinitely many pairwise non-isomorphic $\mathfrak{p}$-primary ideals of multiplicity e. However, we show that for $e \leq 4$ there is a finite characterization of the linear, quadric and cubic generators of all such $\mathfrak{p}$-primary ideals. We apply our results to improve bounds on the projective dimension of ideals generated by three cubic forms.

1. INTRODUCTION

Let S denote a polynomial ring over an algebraically closed field k. Let $x, y \in S$ denote linearly independent linear forms so that $\mathfrak{p} = (x, y)$ is a height two prime ideal with multiplicity $e(S/\mathfrak{p}) = 1$. It has long been known that all height 2 prime (in fact, unmixed) ideals of multiplicity 1 are of this form. (See e.g. [4, p. 112].) Engheta showed that there are essentially two distinct types of $\mathfrak{p}$-primary ideals of multiplicity 2:

Proposition 1.1 (Engheta [5, Proposition 11]). *Suppose that I is a homogeneous $\mathfrak{p}$-primary ideal with multiplicity $e(S/I) = 2$. Then, after possibly a linear change of variables, I has one of the following forms:*

(1) $\big(x, y^2\big)$

(2) $\big(x, y\big)^2 + \big(ax + by\big)$, *where a, b are homogeneous forms of the same degree and* $\operatorname{ht}(x, y, a, b) = 4$.

In particular, $\operatorname{pd}(S/I) \leq 3$ *for all such ideals.*

This result was employed repeatedly in Engheta's proof [5], [7] that if J is an ideal generated by 3 cubic forms, then the projective dimension of S/J is at most 36, a theorem which gave a positive answer to a specific case of Stillman's Question [16, Problem 3.14]. However, the largest known projective dimension of an ideal generated by 3 cubic forms is only 5. See [15] or [8] for surveys about Stillman's Question.

In [9], Huneke et. al. showed that for each $e \geq 3$ and for any $n \in \mathbb{N}$, there exists a $\mathfrak{p}$-primary I with $e(S/I) = e$ and $\operatorname{pd}(S/I) \geq n$. (Similar results are given for ideals primary to linear primes of height 3 or more.) Therefore, no natural extensions of Proposition 1.1 are possible. However, the ideals constructed in [9] have many generators in very large degree, so it seems natural to ask if a finite classification is possible if one further imposes an upper bound on the degrees of the generators of I. Such a result should dramatically improve the projective dimension estimates used in Engheta's papers. Our main results, Theorems 2.1 and 3.1, offer this type of partial classification for $\mathfrak{p}$-primary ideals of multiplicities 3 and 4. The proofs are inductive in nature and rely heavily on the Buchsbaum-Eisenbud acyclicity theorem. We then improve Engheta's upper bound on the projective dimension of an ideal generated by 3 cubics to just 5 in the cases when the unmixed part of, or a direct link of, that ideal is $\mathfrak{p}$-primary. A complete proof that 5 is the optimal upper bound in all cases (which has been conjectured for several years) will appear in a future paper. A similar but less comprehensive strategy was employed by Huneke, Seceleanu, and the authors in [10] to produce a tight bound on the projective dimension of an ideal generated by four quadrics.

Beyond the applications to Stillman's Question, we hope these results will be of interest to algebraic geometers studying vector bundles and multiple or nilpotent structures. Manolache [12] gave a characterization of Cohen-Macaulay scheme structures of multiplicity up to 4 on a codimension 2 linear subspace of the projective space over an algebraically closed field k. These correspond to homogeneous $\mathfrak{p}$-primary ideals that are Cohen-Macaulay when localized at any homogeneous non-maximal prime ideal, that is, Cohen-Macaulay on the punctured spectrum. Our results are a generalization of his in that we do not make any Cohen-Macaulayness assumption. Our methods though are completely different. Manolache exploits connections with vector bundles while our results rely on free resolutions and algebraic arguments. Multiple structures have also been shown by Vatne [19] to be important in the study of Hartshorne's conjecture. See [13] for a survey on nilpotent schemes.

The rest of the paper is organized as follows: In Sections 2 and 3, we give the classification results for multiplicity 3 and multiplicity 4 $\mathfrak{p}$-primary ideals, respectively. In Section 4, we prove that the ideals in the previous two sections are $\mathfrak{p}$-primary, construct their free resolutions and those of ideals directly linked to them. In the final Section 5, we apply our results to show that the projective dimension of S/I, where I is generated by 3 cubic forms, is at most 5 when the unmixed part of I, or an ideal directly linked to it, is $\mathfrak{p}$-primary.

Throughout this paper, k denotes an algebraically closed field, unless otherwise stated. We denote by S a polynomial ring over k without fixing the number of variables. We denote by x and y linearly independent linear forms and set $\mathfrak{p} = (x, y)$. The symbols $a, b, c, \dots$ will denote arbitrary homogeneous forms, with degrees specified at times. We reserve $\alpha, \beta, \gamma, \dots$ for homogeneous degree 0 elements, i.e. elements of k.

2. Multiplicity 3 (x, y)-Primary Ideals

In this section we give a characterization of multiplicity 3 ideals primary to a linear prime $\mathfrak{p} = (x, y)$. While the main result in [9] shows that no finite characterization exists, we compute here the linear, quadric, and cubic parts of all such ideals. The following theorem is thus a best case answer to Engheta's question [5, p. 724] as to whether there is a finite classification of height two unmixed ideals of multiplicity 3 (the non-primary ideals being intersections of known ideals of smaller multiplicity). Here $I_{\geq 4} = \oplus_{i=4}^{\infty} I_i$ denotes the ideal generated by the homogeneous components of I of degree at least 4. By convention we set the unit ideal (1) to have infinite height. We denote by $e(M)$ the multiplicity of a graded module M. In the proofs of Theorems 2.1 and 3.1 we repeatedly use the fact that if $I \subseteq J$ are unmixed ideals of the same height and multiplicity, then $I = J$.

Theorem 2.1. *Let S be a polynomial ring over an algebraically closed field k. Let $x, y \in S$ be independent linear forms and set $\mathfrak{p} = (x, y)$. Suppose that I is a homogeneous $\mathfrak{p}$-primary ideal with $e(S/I) = 3$. Then, after possibly a linear change of variables, I has one of the following forms:*

(i) $\big(x, y\big)^2$
(ii) $\big(x, y^3\big)$
(iii) $\big(x^2, xy, ax + y^2\big)$, *where* $\operatorname{ht}(x, y, a) = 3$
(iv) $\big(x^2, xy, y^3, ax + by^2\big)$, *where* $\operatorname{ht}(x, y, a, b) = 4$
(v) $\big(x, y\big)^3 + \big(ax + by\big)$, *where* $\operatorname{ht}(x, y, a, b) = 4$
(vi) $\big(x, y\big)^3 + \big(x, y\big)\big(ax + by\big) + \big(c(ax + by) + dx^2 + exy + fy^2\big)$, *where* $a, b, c, d, e, f \in S_1$ *such that* $\operatorname{ht}(x, y, a, b) = 4$ *and* $\operatorname{ht}(x, y, c, b^2d - abe + a^2f) = 4$
(vii) $\big(x, y\big)^3 + \big(x, y\big)\big(ax + by\big) + \big(a(ax + by) + cxy + dy^2, b(ax + by) - cx^2 - dxy\big)$, *where* $a, b, c, d \in S_1$ *such that* $\operatorname{ht}(x, y, a, b) = 4$ *and* $\operatorname{ht}(x, y, a, b, c, d) \geq 5$
(viii) $\big(x, y\big)^3 + \big(x, y\big)\big(ax + by\big) + I_{\geq 4}$, *where* $\operatorname{ht}(x, y, a, b) = 4$. *Furthermore, all such ideals are contained in* $\big(x, y\big)^2 + \big(ax + by\big)$.

Moreover, in cases (i) - (vii) we have $\operatorname{pd}(S/I) \leq 4$ *and* $\operatorname{pd}(S/L) \leq 3$ *for some ideal directly linked to I.*

Proof. First note that all the ideals listed in cases *(i)* through *(vii)* are $\mathfrak{p}$-primary with multiplicity 3; this is obvious for *(i)* and *(ii)*; cases *(iii)* through *(vii)* follow from Lemmas 4.3 (twice),4.2, 4.4, and 4.5, respectively. The projective dimension statements follow as well. Therefore, here it remains to show that this is a complete list.

The ideal $J = I : \mathfrak{p}$ is again $\mathfrak{p}$-primary and satisfies $e(S/J) \leq 2$. If $e(S/J) = 1$, then $J = (x, y)$. In this case, $\mathfrak{p}J = \big(x, y\big)^2 \subseteq I$ is an unmixed height 2 ideal of multiplicity 3. Hence $I = \big(x, y\big)^2$, and we are in case *(i)*.

If $e(S/J) = 2$, then by Proposition 1.1, J is one of the following:

(1) $J = \big(x, y^2\big)$, or

(2) $J = \big(x,y\big)^2 + \big(ax+by\big)$, where a,b are forms of the same degree and $\operatorname{ht}(x,y,a,b) = 4$.

Case (1): $J = \big(x, y^2\big)$

We have the containments $\mathfrak{p}J = \big(x^2, xy, y^3\big) \subseteq I \subseteq J = \big(x, y^2\big)$. Because $e\big(S/(x^2,xy,y^3)\big) = 4$, there exists an additional generator $g \in I$, which we take to be of the smallest possible degree. The inclusions above imply that we may take g to have the form $ax + by^2$ with $\operatorname{ht}(x,y,a,b) \geq 3$.

If $\operatorname{ht}(x,y,a,b) > 3$ and $a \notin k$, then $\big(x^2, xy, y^3, ax+by^2\big)$ is unmixed and of multiplicity 3 by Lemma 4.3. So $I = \big(x^2, xy, y^3, ax+by^2\big)$ as in case *(iii)* or case *(iv)* if $b \in k$ and $b \neq 0$. If a is a unit then $b = 0$ (because g is homogeneous) and after possibly a linear change of variables we may assume $a = 1$ so we are in case *(ii)*; if $a = 0$, then b is a unit, and thus we may assume $b = 1$ and we are in case *(i)*.

We now show that $\operatorname{ht}(x,y,a,b) \neq 3$. To the contrary, if this is the case, then a and b share a common factor h of positive degree modulo $\mathfrak{p}$. Then $g = h(a_1x + b_1y^2) + F$ for some $a_1, b_1 \in S$ and $F \in \big(x^2, xy, y^3\big) \subseteq I$. Thus $h(a_1x + b_1y^2) \in I$, and since I is $\mathfrak{p}$-primary and $h \notin \mathfrak{p}$ we obtain $a_1x + b_1y^2 \in I$. This contradicts the minimality of the degree of g.

Case (2): $J = \big(x,y\big)^2 + \big(ax+by\big)$, with $\operatorname{ht}(x,y,a,b) = 4$

We have $\mathfrak{p}J = \big(x,y\big)^3 + \big(x,y\big)\big(ax+by\big) \subseteq I \subseteq \big(x,y\big)^2 + \big(ax+by\big) = J$. Observe that $\mathfrak{p}J = \big(x,y\big)^3 + \big(x,y\big)\big(ax+by\big)$ is $\mathfrak{p}$-primary with multiplicity 4 and local Hilbert function $H_{(S/(\mathfrak{p}J))_\mathfrak{p}} = (1,2,1)$, e.g. by Lemma 4.12. In particular, the Hilbert function of $(S/I)_\mathfrak{p}$ is either $H_{(S/I)_\mathfrak{p}} = (1,1,1)$ or $H_{(S/I)_\mathfrak{p}} = (1,2,0)$. However, the latter case cannot hold, since otherwise $I_\mathfrak{p} = p_\mathfrak{p}^2$ and thus, since both I and $\mathfrak{p}$ are $\mathfrak{p}$-primary, $I = \mathfrak{p}^2$, yielding $J = I : \mathfrak{p} = \mathfrak{p}$, a contradiction. So $\mathfrak{p}_\mathfrak{p}^2 \neq I_\mathfrak{p}$, and I contains an additional generator g of smallest degree that locally is part of a minimal generating set of $\mathfrak{p}_\mathfrak{p}$ and thus $I_\mathfrak{p} = \big(\mathfrak{p}^3 + (g)\big)_\mathfrak{p}$. Write $g = c(ax+by) + dx^2 + exy + fy^2$ for some $c \notin \mathfrak{p}$. First, assume $\deg(g) \geq 4$, we show that $[I]_3 = [pJ]_3$, i.e. I has no other minimal generators of degree at most 3, proving that we are in case *(viii)*. In fact, let $h \in I \setminus \mathfrak{p}J$ with $\deg(h) \leq 3$. By the above we can write $h = c_1(ax+by) + H$ for some $H \in \mathfrak{p}^2$ and by minimality of g, we may further assume $c_1 = 0$, i.e. $h = H \in \mathfrak{p}^2$. Now, observe that $h \in I_\mathfrak{p} \cap \mathfrak{p}_\mathfrak{p}^2 = \big(\mathfrak{p}^3 + (g)\big)_\mathfrak{p} \cap \mathfrak{p}_\mathfrak{p}^2 = \big(\mathfrak{p}^3 + \mathfrak{p}g\big)_\mathfrak{p} = \big(\mathfrak{p}^3 + \mathfrak{p}(ax+by)\big)_\mathfrak{p} = \mathfrak{p}_\mathfrak{p}J_\mathfrak{p}$. Since $\mathfrak{p}J$ is $\mathfrak{p}$-primary, then $h \in \mathfrak{p}J$ – a contradiction.

We may then assume $g = c(ax+by) + dx^2 + exy + fy^2$, $c \notin \mathfrak{p}$, and $\deg(g) \leq 3$. If $\deg(c) = 0$, then by setting $a' = ca + dx + ey$ and $b' = cb + fy$, the ideal I has the form $(x,y)^3 + (a'x + b'y)$ with $\operatorname{ht}(x,y,a',b') = 4$ which is case *(v)*.

Finally we consider the case when $\deg(c) = 1$ and $\operatorname{ht}(x,y,c) = 3$ and so $\deg(g) = 3$. In particular, a,b,d,e,f also have degree 1. If $\operatorname{ht}(x,y,c,b^2d - abe + a^2f) = 4$, then I must be as in case *(vi)* by Lemma 4.4. So we may assume $\mathrm{ht}(x,y,c,b^2d-abe+a^2f)=3$, which is equivalent to saying that the image of $F:=b^2d-abe+a^2f$ is 0 in the polynomial ring $S/(x,y,c)$. If $\mathrm{ht}(x,y,a,b,c)=5$, we have $d,e,f\in(x,y,a,b,c)$. Since $\mathfrak{p}^3\subseteq I$, by modifying d,e,f modulo $\mathfrak{p}$ we may assume $d,e,f\in(a,b,c)$, and clearly we still have $F=0$ modulo (x,y,c). Now, after possibly modifying d,e,f modulo (x,y,c) and modifying a,b modulo (x,y), which preserves I and J, we may assume $d,e,f\in(a,b)$ and $F=0$ still holds in $S/(x,y,c)$. Then $F=0$ in S because F is a cubic in $(a,b)^3\cap(x,y,c)=(a,b)^3(x,y,c)$. Then $a^2g=(ax+by)(aey+adx-bdy)+a^2c(ax+by)\in I$ and since $(x,y)(ax+by)\subseteq I$, then $a^2c(ax+by)\in I$. Since $a^2c\notin p$, it follows that $ax+by\in I$, contradicting the minimality of $\deg(g)$.

We may then assume that additional cubic generators have the form $c(ax+by)+dx^2+exy+fy^2$, where $\mathrm{ht}(x,y,a,b,c)=4$ and $\mathrm{ht}(x,y,c)=3$. After a linear change of variables, we may assume that $a=c$. (Write $c=\alpha x+\beta y+\gamma a+\delta b$, we first set $d'=d+\alpha a$ and $e'=e+\alpha b+\beta a$ and $f'=f+\beta b$. We may then assume $\alpha=\beta=0$. Now $c=\delta a+\gamma b$ and without loss of generality $\gamma\neq 0$. Then set $x=\gamma x'$ and $y=\delta x+y'$.) Since $b^2d=0$ in $S/(x,y,a)$ and since $\mathrm{ht}(x,y,a,b)=4$, we have $d\in(x,y,a)$, that is $d=\alpha x+\beta y+\gamma a$ for some $\alpha,\beta,\gamma\in k$. Therefore

$$\begin{aligned}&a(ax+by)+(e-\gamma b)xy+fy^2\\ &=(a(ax+by)+(\alpha x+\beta y+\gamma a)x^2+exy+fy^2)-\beta x^2y-\gamma x(ax+by)\in I\end{aligned}$$

and so we may assume $d=0$ and $a(ax+by)+exy+fy^2\in I$. Then

$$\begin{aligned}&a\big(b(ax+by)-ex^2-fxy\big)\\ &=b\big(a(ax+by)+exy+fy^2\big)-(ex-fy)(ax+by)\in I.\end{aligned}$$

Since $a\notin\mathfrak{p}$, we have $b(ax+by)-ex^2-fxy\in I$.

If $\mathrm{ht}(x,y,a,b,e,f)\geq 5$, then $\big(x,y\big)^3+\big(x,y\big)\big(ax+by\big)+\big(a(ax+by)+exy+fy^2,b(ax+by)-ex^2-fxy\big)$ is unmixed by Lemma 4.5 and hence equal to I, leaving us in case *(vii)*.

Otherwise, $\mathrm{ht}(x,y,a,b,e,f)\leq 4$, and without loss of generality we may assume that $e,f\in(a,b)$, thus $s=a(ax+by)+ay\ell_1+by\ell_2\in I$ for some $\ell_1,\ell_2\in\mathfrak{p}$. Then $s-\ell_2(ax+by)=a(x(a-\ell_2)+y(b+\ell_1))=a(a'x+b'y)\in I$, and then $a'x+b'y\in I$ where $a'=a-\ell_2$ and $b'=b+\ell_1$, and we are in case *(v)* again. This completes the proof. □

Note that all other $\mathfrak{p}$-primary ideals of multiplicity 3 fall into case *(viii)*. By [9] there are infinitely many such ideals. For example

$$\begin{aligned}I=&\big(x,y\big)^3+\big(x,y\big)\big(ax+by\big)\\ &+\big(a^2(ax+by)-c^2y^2,ab(ax+by)+c^2xy,b^2(ax+by)-c^2x^2\big)\end{aligned}$$

is a $\mathfrak{p}$-primary ideal in $S=k[x,y,a,b,c]$ that falls into case *(viii)*.

We also remark that the hypothesis that the field is algebraically closed is necessary to apply Engheta's result 1.1.

3. Multiplicity 4 (x,y)-Primary Ideals

In this section we give the the classification of low degree generators of all multiplicity 4 ideals primary to a linear prime $\mathfrak{p}=(x,y)$. Necessarily this is a much longer list than that of Theorem 2.1.

Theorem 3.1. *Let S be a polynomial ring over an algebraically closed field k. Let $x,y \in S$ be independent linear forms and set $\mathfrak{p}=(x,y)$. Suppose that I is a homogeneous $\mathfrak{p}$-primary ideal with $e(S/I)=4$. Then, after possibly a linear change of variables, I has one of the following forms:*

(i) $\big(x,y^4\big)$
(ii) $\big(x^2,y^2\big)$
(iii) $\big(x^2,y^2+xy\big)$
(iv) $\big(x^2,xy,y^3\big)$
(v) $\big(x^2,y^2+ax\big)$, *where* $\operatorname{ht}(x,y,a)=3$
(vi) $\big(x^2,xy,y^3+ax\big)$, *where* $\operatorname{ht}(x,y,a)=3$
(vii) $\big(x^2,xy,y^4,ax+by^3\big)$, *where* $\operatorname{ht}(x,y,a,b)=4$
(viii) $\big(x^2,xy^2,y^3+cxy,axy+b(y^2+cx)\big)$, *where* $c\in S_1$ *and* $\operatorname{ht}(x,y,a,b)=4$
(ix) $\big(x^2,xy^2,y^4,ax+by^2\big)$, *where* $\operatorname{ht}(x,y,a,b)=4$
(x) $\big(x^3,xy,y^3,ax^2+by^2\big)$ *where* $\operatorname{ht}(x,y,a,b)=4$
(xi) $\big(x,y\big)^4+\big(ax+by\big)$ *with* $\operatorname{ht}(x,y,a,b)=4$
(xii) $\big(x,y\big)^3+\big(ax^2+bxy+cy^2,dx^2+exy+fy^2\big)$, *where* $\operatorname{ht}\left(x,y,ae-bd,af-cd,bf-ce\right)=4$
(xiii) $\big(x,y\big)^3+\big(axy-by^2,ax^2-cy^2,bx^2-cxy\big)$, *where* $\operatorname{ht}(x,y,a,b,c)=5$
(xiv) $\big(x,y\big)^4+\big(x,y\big)\big(ax+by\big)+\big(c(ax+by)+dx^3+ex^2y+fxy^2+gy^3\big)$, *where* $\operatorname{ht}(x,y,a,b)=4$ *and* $\operatorname{ht}(x,y,c,ga^3-fa^2b+eab^2-db^3)=4$
(xv) $\big(x,y\big)^4+\big(x,y\big)\big(ax+by\big)+\big(a(ax+by)+cx^2y+dxy^2+ey^3,b(ax+by)-cx^3-dx^2y-exy^2\big)$, *where* $\operatorname{ht}(x,y,a,b)=4$ *and* $\operatorname{ht}(x,y,a,b,c,d,e)\geq 5$
(xvi) $\big(x,y\big)^4+\big(x,y\big)\big(ax+by\big)+\big(b(ax+by)+ay^2,x(ax+by)-y^3\big)$, *where* $\operatorname{ht}(x,y,a,b)=4$
(xvii) $\big(x^2,xy^2\big)+I_{\geq 4}$
(xviii) $\big(x^2,xy^2,y^3+axy\big)+I_{\geq 4}$, *where* $\operatorname{ht}(x,y,a)=3$
(xix) $\big(x,y\big)\big(ax+by\big)+I_{\geq 4}$
(xx) $\big(f\big)+I_{\geq 4}$, *where* $\deg(f)=3$
(xxi) $\big(x,y\big)^3+I_{\geq 4}$
(xxii) $\big(x,y\big)^3+\big(ax^2+bxy+cy^2\big)+I_{\geq 4}$, *where* $\operatorname{ht}(x,y,a,b,c)\geq 4$
(xxiii) *All generators of I have degree at least 4.*

Proof. That the ideals in *(i)* through *(v)* are $\mathfrak{p}$-primary of multiplicity 4 is clear. The same statements for cases *(vi)* through *(x)* and *(xii)* through *(xvi)* follow from Lemmas 4.6 to 4.15. Case *(xi)* follows from [5, Lemma 10]. Once again it remains to show that the above list is complete.

Let I be $\mathfrak{p}$-primary with $e(S/I) = 4$. Set $J = I : \mathfrak{p}$. Then $e(S/J) = 2$ or 3. If $e(S/J) = 2$, then, by Proposition 1.1, either $J = \big(x, y^2\big)$ or $J = \big(x, y\big)^2 + \big(ax + by\big)$, where $\operatorname{ht}(x, y, a, b) = 4$.

If $J = \big(x, y^2\big)$, then we have $\mathfrak{p}J = \big(x^2, xy, y^3\big) \subseteq I$. Since $\big(x^2, xy, y^3\big)$ is $\mathfrak{p}$-primary and has multiplicity 4, then $\mathfrak{p}J = I$ putting us in case *(iv)*.

If $J = \big(x, y\big)^2 + \big(ax + by\big)$, then $\mathfrak{p}J = \big(x, y\big)^3 + \big(x, y\big)\big(ax + by\big) \subseteq I$. By Lemma 4.12 the ideal on the left is unmixed of height 2 and multiplicity 4; hence we have equality and we are in case *(xii)*.

We may then assume that $e(S/J) = 3$, and J is one of the ideals from Proposition 2.1.

Case 1: $J = \big(x, y\big)^2$

Since $\mathfrak{p}J = \mathfrak{p}^3 \subseteq I \subseteq \mathfrak{p}^2$ and $\mathfrak{p}J$ has multiplicity 6, we need at least 2 additional linearly independent minimal generators. If all other minimal generators have degree at least 4, then we are in case *(xxi)*. Since k is algebraically closed, if I contains two linearly independent quadrics in $\mathfrak{p}^2$, at least one of them must be a square; after a linear change of coordinates, we may assume it is of the form x^2. The second quadric, is either another square, which we may assume is y^2 (case *(ii)*), a product of the form xy (case *(iv)*), or a product of the form $y(y + \alpha x)$ for some $0 \neq \alpha \in k$. If $\operatorname{char}(k) \neq 2$, we may reduce case to case *(ii)* by completing the square and performing a linear change of variables, but this is not possible if $\operatorname{char}(k) = 2$ and $\alpha = 1$, leaving us in case *(iii)*.

If I only contains one quadric, we may assume it is of the form x^2 or xy. In the first case one has $\big(x^2, xy^2, y^3\big) \subseteq I$. Since I contains no other quadric generators, it must contain an element of the form $axy + by^2 \notin \big(x^2, xy^2, y^3\big)$, which we may take to have minimal degree. If $\operatorname{ht}(x, y, a, b) < 4$, then a and b have a common factor modulo $\mathfrak{p}$. After factoring this out we find an element $a'xy + b'y^2 \in I$ of smaller degree, a contradiction. Thus, we have $\operatorname{ht}(x, y, a, b) = 4$ which leaves us in case *(viii)* (with $c = 0$).

If $xy \in I$, then $\big(x^3, xy, y^3\big) \subseteq I$, and then we may assume there is an element in I of the form $ax^2 + by^2 \notin (x^3, xy, y^3)$, and we may take it to be of minimal degree. As before, one cannot have $\operatorname{ht}(x, y, a, b) < 4$; thus $\operatorname{ht}(x, y, a, b) = 4$ and we are in case *(x)*.

We may now assume that $\big(x, y\big)^3 \subseteq I \subseteq \big(x, y\big)^2$ and that I contains no quadrics. If I contains only one more cubic up to linear dependence, we may take it in the form $ax^2 + bxy + cy^2$ with $a, b, c \in S_1$. If $\operatorname{ht}(x, y, a, b, c) = 3$, then, after factoring out the common factor modulo $\mathfrak{p}$, we find that I contains a quadric, a contradiction. Hence $\operatorname{ht}(x, y, a, b, c) \geq 4$ and we are in case *(xxii)*. Therefore, we may assume I contains two linearly independent cubics of the form $ax^2 + bxy + cy^2, dx^2 + exy + fy^2$ not lying in $\mathfrak{p}^3$. If $\operatorname{ht}(x, y, ae - bd, af - cd, bf - ce) = 4$, then we are in case *(xii)*.

We may now assume that $\operatorname{ht}(x, y, ae-bd, af-cd, bf-ce) \leq 3$, and define $\mathbf{M}$ to be the matrix

$$\mathbf{M} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}.$$

Since $ax^2+bxy+cy^2, dx^2+exy+fy^2 \notin \mathfrak{p}^3$ and are linearly independent, $\operatorname{ht}\big((x,y)+I_2(\mathbf{M})\big) = 3$. Since $\mathfrak{p}^3 \subseteq I$, we can reduce a,b,c,d,e,f modulo $\mathfrak{p}$ to assume $\operatorname{ht}(I_2(\mathbf{M})) = 1$ in S. Then $I_2(\mathbf{M})$ is contained in a principal ideal, either generated by a linear form $\ell \notin \mathfrak{p}$ or a quadric form $q \notin \mathfrak{p}$. In the latter case, then writing Δ_i for the minor of $\mathbf{M}$ omitting column i, we have

$$\begin{aligned} s &= (d+e+f)(ax^2+bxy+cy^2) - (a+b+c)(dx^2+exy+fy^2) \\ &= (\Delta_2+\Delta_3)x^2 + (\Delta_1+\Delta_3)xy + (\Delta_1+\Delta_2)y^2 \in q\mathfrak{p}^2 \cap I \end{aligned}$$

Thus $s = qF \in I$ for some nonzero quadric $F \in (x,y)^2$ and since $q \notin \mathfrak{p}$, then the quadric $F \in I$, a contradiction.

If $I_2(\mathbf{M}) \subseteq (\ell)$, then there are elements $b', c' \in S$ such that $b'xy + c'y^2 \in I$; indeed, if $\operatorname{ht}(a,d) \leq 1$, then a linear combination of $g_1 := ax^2+bxy+cy^2$ and $g_2 := dx^2+exy+fy^2$ has the form $b'xy+c'y^2$. If $\operatorname{ht}(a,d) = 2$, then

$$t = d(ax^2+bxy+cy^2) - a(dx^2+exy+fy^2) = xy\Delta_3 + y^2\Delta_2 = \ell b'xy + \ell c'y^2 \in I.$$

Observe that $t \neq 0$, since otherwise $\Delta_3 = \Delta_2 = 0$, and then $b, c \in (a)$ and $e, f \in (d)$, yielding again that I contains a quadric, a contradiction. Since $t \neq 0$ and $\ell \notin \mathfrak{p}$ then $0 \neq b'xy + c'y^2 \in I$. Similarly, we may find a generator of I of the form $d'x^2 + e'xy$; after replacing g_1 and g_2 by these two generators, we may assume that $\mathbf{M}$ has the form

$$\mathbf{M} = \begin{pmatrix} 0 & b' & c' \\ d' & e' & 0 \end{pmatrix},$$

and $\operatorname{ht}(I_2(\mathbf{M})) = \operatorname{ht}(b'd', c'd', c'e') = 1$. If $\operatorname{ht}(b',c') = 1$, or $\operatorname{ht}(d',e') = 1$, then I will contain a quadric upon factoring out the common linear factor, a contradiction. Hence we must have $\operatorname{ht}(c',d') = 1$ and we may assume $c' = d'$. Then $e'(b'xy+c'y^2) - b'(c'x^2+e'xy) = c'(-b'x^2+e'y^2) \in I$ and thus $-b'x^2+e'y^2 \in I$. If $\operatorname{ht}(b',c',e') \leq 2$, then we may factor a linear form out of one of these three generators and again find a quadric in I, a contradiction. Thus $\operatorname{ht}(b',c',e') = 3$ and then we are in case *(xiii)*.

Case 2: $J = \big(x, y^3\big)$

In this case, $\mathfrak{p}J = \big(x^2, xy, y^4\big) \subseteq I \subseteq \big(x, y^3\big) = J$. If I contains a linear form, then $I = \big(x, y^4\big)$, which is case *(i)*. If I contains no linear forms, then I contains no additional quadric form q either, since otherwise we may take q to be in the form $q = ax$ with $a \notin \mathfrak{p}$and then $x \in I$. We assume then that I contains an additional cubic form, which we may take to be in the form $\alpha y^3 + ax$ with $\alpha \in k$ and $a \in S_2$. If $\alpha = 0$, we again have $x \in I$, so we may assume $\alpha = 1$. Since $\big(x^2, xy\big) \subseteq I$, after possibly reducing a modulo $\mathfrak{p}$ we may assume either $a = 0$, or $\operatorname{ht}(x,y,a) = 3$. In the first situation $I = \big(x^2, xy, y^3\big)$, which is impossible because then $J = \big(x, y^2\big)$; in the second case $I = \big(x^2, xy, y^3 + ax\big)$ so we are in case *(vi)*. Finally, if I

contains no additional forms of degree at most three, then I must contain an element of the form $ax + by^3$, which we may be take to have minimal degree. If $\mathrm{ht}(x, y, a, b) = 3$, then a and b have a common factor modulo $\mathfrak{p}$. After factoring out, we find a term of smaller degree in I, a contradiction. Hence $\mathrm{ht}(x, y, a, b) = 4$, so I is as in case *(vii)*.

Case 3: $J = \left(x^2, xy, ax + y^2\right)$ with $\mathrm{ht}(x, y, a) = 3$

In this case $\mathfrak{p}J = (x^3, x^2y, xy^2, ax^2, y^3 + axy) \subseteq I \subseteq \left(x^2, xy, ax + y^2\right) = J$. Since $ax^2 \in I$, $a \notin \mathfrak{p}$ and I is $\mathfrak{p}$-primary, $x^2 \in I$, and thus $\left(x^2, xy^2, y^3 + axy\right) \subseteq I$. If I contains an additional quadric, it must have the form $\alpha xy + \beta(y^2 + ax)$ for some $\alpha, \beta \in k$. If $\beta \neq 0$, we may assume $\beta = 1$, thus $I = (x^2, y^2 + a'x)$, where $a' = a + \alpha y$, which is case *(v)*. If $\beta = 0$, then $I = \left(x^2, xy, y^3\right)$; thus we would have $J = I : \mathfrak{p} = \left(x, y^2\right)$, a contradiction.

We may now assume I contains no quadrics but at least an additional cubic,which we may take of the form $bxy + c(y^2 + ax)$ with $b, c \in S_1$. If $\mathrm{ht}(x, y, b, c) = 4$, then we are in case *(viii)*. If not, then b and c have a common factor modulo $\mathfrak{p}$; factoring out yields a quadric in I, a contradiction. Finally, if I contains no additional quadrics or cubics, we are in case *(xviii)*.

Case 4: $J = \left(x^2, xy, y^3, ax + by^2\right)$ with $\mathrm{ht}(x, y, a, b) = 4$

In this case $\mathfrak{p}J = \left(x^3, x^2y, xy^2, y^4, axy + by^3, ax^2\right) \subseteq I$ and, similarly to the above, this yields $\left(x^2, xy^2, y^4, axy + by^3\right) \subseteq I \subseteq \left(x^2, xy, y^3, ax + by^2\right) = J$. If I contains an additional quadric form, then $I = \left(x^2, xy, y^3\right)$, which is impossible because then $J = \left(x, y^2\right)$. Assume then I contains no additional quadrics but at least an additional cubic form. If $\deg(a) = 2$ and $\deg(b) = 1$, then we may take this cubic of the form $cxy + \alpha y^3 + \beta(ax + by^2)$. Now, if $\beta \neq 0$, then we may assume $\beta = 1$. Replacing a by $a' = a + cy$ and b by $b' = b + \alpha y$, one obtains that $a'x + b'y^2 \in I$; since $\left(x^2, xy^2, y^4, a'x + b'y^2\right) \subseteq I$ and $\mathrm{ht}(x, y, a', b') = 4$, we are in case *(ix)*. If $\beta = 0$ and $\alpha \neq 0$, we may assume $\alpha = 1$ and using that $axy + by^3 \in I$ one sees that $(a - bc)y^3 \in I$. Since $a - bc \notin \mathfrak{p}$, we have $y^3 \in I$; thus $I = \left(x^2, xy, y^3\right)$, which is again a contradiction.

If $\alpha = \beta = 0$ or if $\deg(a) \geq 3$, then any additional cubic in I has the form cxy, with $c \in S_1$ and $c \notin \mathfrak{p}$. It follows that $xy \in I$ and thus $I = (x^2, xy, y^3)$, giving again a contradiction.

Finally, if I contains no additional quadric or cubic forms, we are in case *(xvii)*.

Case 5: $J = \left(x, y\right)^3 + \left(ax + by\right)$ with $\mathrm{ht}(x, y, a, b) = 4$

In this case, $\mathfrak{p}J = \left(x, y\right)^4 + \left(x, y\right)\left(ax + by\right) \subseteq I \subseteq \left(x, y\right)^3 + \left(ax + by\right) = J$. If $ax + by \in I$, then we are in case *(xi)*. Hence we may assume that I contains no quadrics.

Observe that if I contains a cubic $s \in \mathfrak{p}^3$, then $\mathfrak{p}^3 \subseteq I$ (because $\mathfrak{p}^3_{\mathfrak{p}} = \left(\mathfrak{p}^2(ax + by) + (s)\right)_{\mathfrak{p}} \subseteq I_{\mathfrak{p}}$) putting us in case *(xii)*.

Now, if I contains an additional cubic form, we may take it to be in the form $g := c(ax + by) + \alpha x^3 + \beta x^2y + \gamma xy^2 + \delta y^3$, where $\alpha, \beta, \gamma, \delta \in k$ and

$\deg(a) = \deg(b) = \deg(c) = 1$, or $\deg(a) = \deg(b) = 2$ and $c \in k$. Notice that by the above argument, if $c = 0$ we are in case *(xii)*. Then we may assume $c \neq 0$. We may also assume $\alpha, \beta, \gamma, \delta$ are not all zero since otherwise $c(ax+by) \in I$ and then $ax+by \in I$. If $\deg(a) = \deg(b) = 2$, then c is a unit and I is as in case *(xiv)*, since $(x, y, c) = S$ and the height condition is met automatically. If $\deg(a) = \deg(b) = \deg(c) = 1$, we may assume that $c \notin \mathfrak{p}$, since otherwise after possibly a linear combination of generators we find one with $c = 0$. Hence $\operatorname{ht}(x, y, c) = 3$. If $\operatorname{ht}(x, y, c, \delta a^3 - \gamma a^2 b + \beta a b^2 - \alpha b^3) = 4$, we are in case *(xiv)*. Otherwise, since $\operatorname{ht}(x, y, a, b) = 4$, we must have $c \in (x, y, a, b)$. We may again reduce to the case $c \in (a, b)$, and after possibly a linear change of the variables a, b, x, y, we may assume $c = b$. Since $\operatorname{ht}(x, y, b, \delta a^3 - \gamma a^2 b + \beta a b^2 - \alpha b^3) = 3$ and $\operatorname{ht}(x, y, a, b) = 4$, we deduce that $\delta = 0$, thus $g = b(ax+by) + \alpha x^3 + \beta x^2 y + \gamma x y^2$. Observe that

$$ag - (\alpha x^2 + \beta xy + \gamma y^2)(ax+by) = b\big(a(ax+by) - \alpha x^2 y - \beta x y^2 - \gamma y^3\big) \in I,$$

and since $b \notin \mathfrak{p}$, then $a(ax+by) - \alpha x^2 y - \beta x y^2 - \gamma y^3 \in I$. Since at least one of α, β, γ is nonzero, we are in case *(xv)*.

Finally, if all additional generators of I have degree at least 4, then we are in case *(xix)* or *(xxiii)*.

Case 6: $J = \big(x, y\big)^3 + \big(x, y\big)\big(ax+by\big) + \big(c(ax+by) + dx^2 + exy + fy^2\big)$ with $\operatorname{ht}(x, y, a, b) = 4$ and $\operatorname{ht}(x, y, c, b^2 d - abe + a^2 f) = 4$

In this case $\mathfrak{p}J = \big(x, y\big)^4 + \big(x, y\big)^2\big(ax+by\big) + \big(x, y\big)\big(c(ax+by) + dx^2 + exy + fy^2\big) \subseteq I \subseteq J$. The ideal on the left has multiplicity 5 and J is generated in degree 3 and higher. So, either I contains at most one linearly independent cubic forms (cases *(xxiii)* and *(xx)*) or I has at least two linearly independent cubic generators of the form

$$s_i = F_i + \ell_i(ax+by) + \alpha_i(c(ax+by) + dx^2 + exy + fy^2),$$

with $\alpha_i \in k$, $\ell_i \in \mathfrak{p}$, and $F_i \in \mathfrak{p}^3$ for $i = 1, 2$. If $\alpha_i \neq 0$ for some i, then after a linear change of variables we may assume $F_1 = \ell_1 = 0$, $\alpha_1 = 1$, and $\alpha_2 = 0$. After a further change of variables we may assume $\ell_2 = x$ or $\ell_2 = 0$. If $\ell_2 = 0$, then $0 \neq F_2 \in \mathfrak{p}^3 \cap I$. It follows that $\mathfrak{p}^3 \subseteq I$ and $\mathfrak{p}^2 \subseteq J$, a contradiction. If $\ell_2 = x$, then

$$cs_2 - x(c(ax+by) + dx^2 + exy + fy^2) = cF_2 - dx^3 - ex^2 y - fxy^2 \in I.$$

Write $F_2 = \beta_1 x^3 + \beta_2 x^2 y + \beta_3 x y^2 + \beta_4 y^3$, where $\beta_i \in k$ for $i = 1, 2, 3, 4$ and consider this element in $I_{\mathfrak{p}}$ along with the elements $x^2(ax+by), xy(ax+by), y^2(ax+by)$. If these are linearly independent in $I_{\mathfrak{p}}$, then $\mathfrak{p}_{\mathfrak{p}}^3 \subset I_{\mathfrak{p}}$. Since I is $\mathfrak{p}$-primary, we would have $\mathfrak{p}^3 \subseteq I$, a contradiction. Therefore

$$\det \begin{pmatrix} c\beta_1 - d & c\beta_2 - e & c\beta_3 - f & \beta_4 \\ a & b & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & b \end{pmatrix} \in \mathfrak{p}.$$

Reducing c, d, e, f module $\mathfrak{p}$ we may assume this determinant is 0. Therefore by linearity of determinants

$$c \cdot \det \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ a & b & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & b \end{pmatrix} = \det \begin{pmatrix} d & e & f & 0 \\ a & b & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & b \end{pmatrix},$$

or equivalently

$$c(-\beta_4 a^3 + \beta_3 a^2 b - \beta_2 a b^2 + \beta_1 b^3) = b(a^2 f - abe + b^2 d).$$

Since $\operatorname{ht}(x, y, a, b) = 4$ and $\operatorname{ht}(x, y, c, a^2 f - abe + b^2 d) = 4$, we may take $b = c$. Moreover, $d, e, f \in (a, b)$. Since $(x, y)^2(ax + by) \subseteq I$, we may assume that $d, e \in (b)$. After another change of variables, we may assume that $d = e = 0$ and $f \in (a)$. If $f = 0$, then $b(ax + by) \in I$ and hence $ax + by \in I$, a contradiction. Therefore we may take $f = a$ and assume $b(ax + by) + ay^2 \in I$. Now we have that

$$b(y^3 - x(ax + by)) = y^2(ax + by) - x(b(ax + by) + ay^2) \in I.$$

Since $b \notin \mathfrak{p}$, we have $y^3 - x(ax + by) \in I$. By Lemma 4.15, I has the form of case *(xvi)*.

We may now assume that $\alpha_1 = \alpha_2 = 0$. If $\operatorname{ht}(\ell_1, \ell_2) \leq 1$, then after taking a linear combination of s_1 and s_2 we may further assume that $\ell_1 = 0$. As above we get $0 \neq F_1 \in \mathfrak{p}^3 \cap I$ and hence $\mathfrak{p}^3 \subseteq I$, a contradiction. If $\operatorname{ht}(\ell_1, \ell_2) = 2$, then after taking a linear combination of s_1 and s_2 we may assume that $\ell_1 = x$ and $\ell_2 = y$. Two determinantal calculations as above imply that $(c) = (a)$ and $(c) = (b)$, contradicting that $\operatorname{ht}(x, y, a, b) = 4$.

Case 7: $J = \big(x, y\big)^3 + \big(x, y\big)\big(ax + by\big) + \big(a(ax + by) + cxy + dy^2, b(ax + by) - cx^2 - dxy\big)$, with $\operatorname{ht}(x, y, a, b) = 4$ and $\operatorname{ht}(x, y, a, b, c, d) \geq 5$

In this case $\mathfrak{p}J = \big(x, y\big)^4 + \big(x, y\big)^2\big(ax + by\big) + \big(x, y\big)\big(a(ax + by) + cxy + dy^2, b(ax + by) - cx^2 - dxy\big) \subseteq I \subseteq J$. The ideal on the left is generated by quartics and has multiplicity 5. If I contains zero or one cubics, we are in cases *(xxiii)* or *(xx)*, respectively.

We now show that I cannot have two or more linearly independent cubics. First we show that I cannot have three linearly independent cubics. Any cubic in I has the form

$$s = F + \ell(ax + by) + \alpha(a(ax + by) + cxy + dy^2) + \beta(b(ax + by) - cx^2 - dxy),$$

for some cubic $F \in \mathfrak{p}^3$ and some linear form $\ell \in \mathfrak{p}$. If there are at least three linearly independent cubics, we may find one with $\alpha = \beta = 0$; then $g = as - \ell\big(a(ax+by)+cxy+dy^2\big) = aF - y\ell(cx+dy) \in I \cap \big(x, y\big)^3$ and, by the height condition, $g \notin (ax+by)_{\mathfrak{p}}$. It follows that $\mathfrak{p}^3_{\mathfrak{p}} \subseteq \big((g)+\mathfrak{p}^2(ax+by)\big)_{\mathfrak{p}} \subseteq I_{\mathfrak{p}}$, thus $\mathfrak{p}^3 \subseteq I$ and then $\mathfrak{p}^2 \subseteq J$, a contradiction.

We now rule out the case where I contains exactly two linearly independent cubics of the above form. After taking linear combinations of those generators, we may assume they have the form

$$s_1 = a(ax+by) + y(cx+dy) + F_1 + \ell_1(ax+by)$$
$$s_2 = b(ax+by) - x(cx+dy) + F_2 + \ell_2(ax+by)$$

where F_1, F_2 are cubics in $\mathfrak{p}^3$ and ℓ_1, ℓ_2 are linear forms in $\mathfrak{p}$; otherwise we have a form as in the previous case. We claim that $\mathfrak{p}_\mathfrak{p}^3 \subseteq I_\mathfrak{p}$, which is again a contradiction. Let $s_3 = s_1(ab - (cx+dy) - b\ell_1 + a\ell_2) - s_2(a^2) \in I$, then modulo $\mathfrak{p}^4 + \mathfrak{p}^2(ax+by)$ we have

$$s_3 = y(\ell_2 a - \ell_1 b - (cx+dy))(cx+dy) + a(bF_1 - aF_2)$$

Observe that $s_3 \notin \mathfrak{p}^2(ax+by)$, since otherwise $-y(cx+dy)^2 \in \mathfrak{p}^2(ax+by) + (a,b)\mathfrak{p}^3$ and then $(cx+dy)^2 \in \mathfrak{p}(ax+by) + (a,b)\mathfrak{p}^2$, which contradicts the height assumption. Thus $s_3 \in \big(I \cap \mathfrak{p}^3\big) \setminus \mathfrak{p}^2(ax+by)$. It follows as above that $\mathfrak{p}_\mathfrak{p}^3 \subseteq I_\mathfrak{p}$, which is impossible.

Case 8: $J = \big(x,y\big)^3 + \big(x,y\big)\big(ax+by\big) + I_{\geq 4}$, with $\operatorname{ht}(x,y,a,b) = 4$

We have $\mathfrak{p}J = \big(x,y\big)^4 + \big(x,y\big)^2\big(ax+by\big) \subseteq I \subseteq \big(x,y\big)^3 + \big(x,y\big)\big(ax+by\big) + I_{\geq 4} = J$. Note also that $I \subseteq J \subseteq \big(x,y\big)^2 + \big(ax+by\big)$. We observe that I does not contain three linearly independent cubics, since otherwise there is a non-zero cubic in $\big(x,y\big)^3 \cap I$, then $\mathfrak{p}_\mathfrak{p}^3 \subseteq I_\mathfrak{p}$ and hence $\mathfrak{p}^3 \subseteq I$ and $\mathfrak{p}^2 \subseteq J$. In particular, we may assume $\deg(a) = \deg(b) = 1$ and that I contains at most two linearly independent cubics. Also, by the above, we may assume I does not contain a nonzero cubic in $\big(x,y\big)^3$. First we assume I contains exactly two linearly independent cubics. After a linear change of variables, we may take them to be of the form $s_1 = x(ax+by)+F_1, s_2 = y(ax+by)+F_2$, where $F_1, F_2 \in \mathfrak{p}^3$. After a linear change of the variables a and b, we may assume that $F_1 = y^3$. Setting $J' = \big(x,y\big)^4 + \big(x,y\big)^2\big(ax+by\big) + \big(s_1, s_2\big) \subseteq I$ one has $H_{(S/J')_\mathfrak{p}} = (1, 2, h_2, h_3)$. Observe that $e(S/J') \geq 5$ since otherwise $h_3 = 0$ and then $\mathfrak{p}_\mathfrak{p}^3 \subseteq J'_\mathfrak{p} \subseteq I_\mathfrak{p}$, which yields $\mathfrak{p}^3 \subseteq I$ and $\mathfrak{p}^2 \subseteq J$, a contradiction. Then I must contain another form (of degree at least 4) in $\big(x,y\big)^2 + \big(ax+by\big)$ but not in J'. Since $\big(x,y\big)^2\big(ax+by\big) \subseteq I$, after rewriting, we may take this form to be $t_1 = c(ax+by) + dx^2 + exy + fy^2$, for some $c \notin \mathfrak{p}$. Hence

$$t_2 := xt_1 - cs_1 = dx^3 + ex^2y + fxy^2 - cy^3 \in I$$

as well. If $t_2 \notin \mathfrak{p}^2(ax+by)$, then $\mathfrak{p}_\mathfrak{p}^3 \subseteq I_\mathfrak{p}$ and hence $\mathfrak{p}^3 \subseteq I$, a contradiction. Therefore $t_2 \in \mathfrak{p}^2(ax+by)$. Set $S = k[x,y,a,b,\underline{z}]$. Since $\mathfrak{p}^2(ax+by) + \mathfrak{p}^4 \subseteq I$, we may assume that $c, d, e, f \in T = k[a,b,\underline{z}]$ and, since $t_2 \in \mathfrak{p}^2(ax+by)$, that $c = -c'b$, $d = e'a$, $e = d'a + e'b$, and $f = c'a + d'b$. Then

$$t_1 = (ax+by)(-bc' + e'x + d'y) + ac'y^2$$

so we find

$$yt_1 + bc's_2 - (e'x + d'y)\big(y(ax+by)\big) = c'(ay^3 + bF_2) \in I.$$

Since $c' \notin \mathfrak{p}$, we have $ay^3 + bF_2 \in I$, but $ay^3 + bF_2 \notin \mathfrak{p}^4 + \mathfrak{p}^2(ax + by)$ and its image in $R_{\mathfrak{p}}$ is in $\mathfrak{p}^3_{\mathfrak{p}}$. It follows that $\mathfrak{p}^3 \subseteq I$, a contradiction.

If I contains only one cubic (up to scalar multiples), we are in case *(xx)*. Finally, if I contains no cubics we are in case *(xxiii)*.

□

Remark 3.2. To show that cases *(xvii)* to *(xxiii)* are necessary, we note that the following are $\mathfrak{p}$-primary ideals of multiplicity 4 in the ring $k[x, y, a, b, c, d, e]$ which do not occur in any of the cases *(i)* to *(xv)*. (This can be checked easily in Macaulay2 [14].)

(xvii) $\big(x^2, xy^2, y^4, a^{24}y^3 + b^{13}(b^{13}x + c^{12}y^2), a^{24}xy - c^{12}(b^{13}x + c^{12}y^2), y(b^{13}x + c^{12}y^2)\big)$

(xviii) $\big(x^2, xy^2, axy + y^3, b^2xy + cy^3 + d^2(y^2 + ax)\big)$

(xix) $\big(x, y\big)^4 + \big(x, y\big)\big(ax + by\big) + \big(a^2e(ax + by) + d^2xy^2 - c^2y^3, abe(ax + by) - d^2x^2y + c^2xy^2, b^2e(ax + by) + d^2x^3 - c^2x^2y\big)$

(xx) $\big(x, y\big)^4 + \big(x, y\big)^2\big(ax+by\big) + \big(x, y\big)\big(b(ax+by) - cx^2\big) + \big(a(ax+by) + cxy\big)$

(xxi) $\big(x, y\big)^3 + \big(bcxy - ady^2, b^2x^2 - d^2y^2, abx^2 - cdxy\big)$

(xxii) $\big(x, y\big)^3 + \big(ax^2 + bxy, (a^2 - c^2)xy + aby^2, c^2x^2 + abxy + b^2y^2\big)$

(xxiii) $\big(x, y\big)^4 + \big(x, y\big)\big(a^2x + b^2y\big) + \big(a^2(a^2x + b^2y) + acx^2y + (bc + ad)xy^2 + bdy^3, ab(a^2x + b^2y) + (bc - ac)x^2y + (bd - ad)xy^2, b^2(a^2x + b^2y) - acx^3 - (bc + ad)x^2y - bdxy^2\big)$

The example for case *(xvii)* is essentially $I_{X_{13}}$ in [12]. The exponents are needed to ensure that the ideal is homogeneous. Note that all cases are disjoint with the exception of case *(xi)* when $\deg(a) = \deg(b) = 2$, which is of the form of case *(xx)*.

4. Resolutions of (x, y)-Primary Ideals

In this section we collect the necessary resolutions and arguments to show that each of the ideals in Theorems 2.1 and 3.1 are indeed (x, y)-primary. We also compute certain ideals directly linked to these ideals along with the multiplicity and projective dimension of each. The argument is essentially the same in every case: for a given ideal I, one resolves S/I generically, say with the help of Macaulay2 [14]. This gives a complex $F_\bullet$ with differential maps ∂_i. By the Buchsbaum-Eisenbud acyclicity theorem [3, Theorem 20.9], $F_\bullet$ is a resolution of S/I when $\mathrm{ht}\,(I_{r_i}(\partial_i)) \geq i$, where $r_i = \sum_{j=i}^{\mathrm{pd}(S/I)} (-1)^{j-i} \mathrm{rank}(F_j)$ denotes the expected rank. If moreover $\mathrm{ht}(I_r(\partial_i)) \geq i + 1$ for $i > \mathrm{ht}(I)$, then I is unmixed; see [9, Proposition 2.4]. Thus it suffices to find a regular sequence of the appropriate length or subideal of a given height in each ideal of minors. While all the details of this section may be checked by hand, for the reader's convenience we have posted a parallel Macaulay2 file at `http://users.rider.edu/~jmccullough/3cubics.m2`.

Throughout this section x, y denote independent linear forms and $a, b, c, etc.$ denote homogeneous elements, not necessarily linear. We set $\mathfrak{p} = (x, y)$.

Lemma 4.1. *If*

$$I = \big(x^3, y^3, x^2y^2, x^2ya + xy^2b, c(a^2x^2 + abxy + b^2y^2) + dx^2y + exy^2\big),$$

where $\operatorname{ht}(x, y, a, b) \geq 4$ *and* $\operatorname{ht}(x, y, c, ae - bd) \geq 4$*, then* I *is* $\mathfrak{p}$*-primary,* $\operatorname{pd}(S/I) = 3$*, and* $e(S/I) = 6$*.*

Proof. Consider the complex

$$S \xleftarrow{\partial_1} S^5 \xleftarrow{\partial_2} S^6 \xleftarrow{\partial_3} S^2 \longleftarrow 0,$$

where

$$\partial_1 = \begin{pmatrix} x^3 & y^3 & ax^2y + bxy^2 & x^2y^2 & a^2cx^2 + abcxy + b^2cy^2 + dx^2y + exy^2 \end{pmatrix},$$

$$\partial_2 = \begin{pmatrix} -ay & 0 & -y^2 & 0 & -a^2c - dy & 0 \\ 0 & bx & 0 & -x^2 & 0 & -b^2c - ex \\ x & -y & 0 & 0 & -bc & -ac \\ -b & a & x & y & -e & -d \\ 0 & 0 & 0 & 0 & x & y \end{pmatrix},$$

$$\partial_3 = \begin{pmatrix} -y & ac \\ -x & bc \\ a & d \\ -b & -e \\ 0 & -y \\ 0 & x \end{pmatrix}.$$

One checks that $I_1(\partial_1) = I$, that $x^5, y^5 \in I_4(\partial_2)$, and $x^2, y^2, a^2c + dy, b^2c - ex, ae - bd \in I_2(\partial_3)$. It follows that this is a resolution of S/I and I is unmixed. Since $\sqrt{I} = \mathfrak{p}$, then I is $\mathfrak{p}$-primary, $(S/I)_\mathfrak{p}$ has Hilbert function $H_{(S/I)_\mathfrak{p}}(1, 2, 2, 1)$, and thus $e(S/I) = 6$. □

Lemma 4.2. *If*

$$I = \big(x, y\big)^3 + \big(ax + by\big),$$

where $\operatorname{ht}(x, y, a, b) = 4$*, then* I *is* $\mathfrak{p}$*-primary,* $\operatorname{pd}(S/I) = 3$*, and* $e(S/I) = 3$*. Moreover,*

$$L = \big(x^3, y^3\big) : I = \big(x^3, x^2y^2, y^3, ax^2y - bxy^2, a^2x^2 - abxy + b^2y^2\big)$$

and $\operatorname{pd}(S/L) = 3$*.*

Proof. That I has the prescribed projective dimension and multiplicity follows from [5, Lemma 10]. Set $L' = \big(x^3, x^2y^2, y^3, ax^2y - bxy^2, a^2x^2 - abxy + b^2y^2\big)$. It is clear that $L' \subseteq (x^3, y^3) : I$. It follows from Lemma 4.1 that L' is $\mathfrak{p}$-primary, $\operatorname{pd}(S/L') = 3$ and $e(S/L') = 6$. Hence $L' = (x^3, y^3) : I$. □

Lemma 4.3. *If*

$$I = \big(x^2, xy, y^3, ax + by^2\big),$$

where $\operatorname{ht}(x, y, a, b) > 3$, *then* I *is* $\mathfrak{p}$*-primary,* $\operatorname{pd}(S/I) = 3$, *and* $e(S/I) = 3$. *Moreover,*

$$L = \big(x^2, y^3\big) : I = \big(x^2, xy, y^3, ax - by^2\big)$$

and $\operatorname{pd}(S/L) = 3$.

Proof. That I is $\mathfrak{p}$-primary, $\operatorname{pd}(S/I) = 3$ and $e(S/I) = 3$ follows from [5, Lemma 12]. Set $L' = (x^2, xy, y^3, ax - by^2)$. Also by [5, Lemma 12], we have that L' is $\mathfrak{p}$-primary, $e(S/L') = 3$ and $\operatorname{pd}(S/L') = 3$. Since $L' \subseteq (x^2, y^3) : I$, it follows that $L' = (x^2, y^3) : I$. □

Lemma 4.4. *If*

$$I = \big(x, y\big)^3 + \big(x, y\big)\big(ax + by\big) + \big(c(ax + by) + dx^2 + exy + fy^2\big),$$

where $\operatorname{ht}(x, y, a, b) = 4$ *and* $\operatorname{ht}(x, y, c, b^2d - abe + a^2f) = 4$, *then* I *is* $\mathfrak{p}$*-primary,* $\operatorname{pd}(S/I) = 3$, *and* $e(S/I) = 3$. *Moreover, if* $L = (x^3, y^3) : I$, *then*

$$L = (x, y)^3 + \big(x^2y^2, ax^2y - bxy^2, c(a^2x^2 - abxy + b^2y^2) - bdx^2y + (be - af)xy^2\big)$$

and $\operatorname{pd}(S/L) = 3$.

Proof. Consider the complex

$$S \xleftarrow{\partial_1} S^7 \xleftarrow{\partial_2} S^9 \xleftarrow{\partial_3} S^3 \longleftarrow 0,$$

where

$$\partial_1 = (acx + dx^2 + bcy + exy + fy^2 \quad ax^2 + bxy \quad x^3 \quad axy + by^2 \quad x^2y \quad xy^2 \quad y^3)$$

$$\partial_2 = \begin{pmatrix} -x & 0 & -y & 0 & 0 & 0 & 0 & 0 & 0 \\ c & -x & 0 & -y & 0 & 0 & 0 & 0 & 0 \\ d & a & 0 & 0 & 0 & -y & 0 & 0 & 0 \\ 0 & 0 & c & x & -x & 0 & -y & 0 & 0 \\ e & b & d & 0 & a & x & 0 & -y & 0 \\ f & 0 & e & 0 & b & 0 & a & x & -y \\ 0 & 0 & f & 0 & 0 & 0 & b & 0 & x \end{pmatrix}$$

$$\partial_3 = \begin{pmatrix} y & 0 & 0 \\ 0 & y & 0 \\ -x & 0 & 0 \\ c & -x & 0 \\ 0 & -x & y \\ d & a & 0 \\ 0 & 0 & -x \\ e & b & a \\ f & 0 & b \end{pmatrix}$$

We note that $I = I_1(\partial_1)$, also $x^6, y^6 \in I_6(\partial_2)$, and $x^3, y^3, a^2c + adx, b^2c + bex - afx, b^2d - abe + a^2f \in I_3(\partial_3)$. Therefore, the above complex is a resolution of S/I and I is unmixed. Since $\sqrt{I} = \mathfrak{p}$ then I is $\mathfrak{p}$-primary, the Hilbert function of $(S/I)_\mathfrak{p}$ is $H_{(S/I)_\mathfrak{p}} = (1, 1, 1)$, thus $e(S/I) = 3$.

Now set $L' = (x,y)^3 + (x^2y^2, ax^2y - bxy^2, c(a^2x^2 - abxy + b^2y^2) - bdx^2y + (be-af)xy^2)$. It is easy to check that $L' \subseteq (x^3, y^3) : I$. Since $\operatorname{ht}(x,y,a,b) = 4$ and $\operatorname{ht}(x,y,c,a(be-af) - b(bd)) = \operatorname{ht}(x,y,c,a^2f - abe + b^2d) = 4$, it follows from Lemma 4.1 that L' is $\mathfrak{p}$-primary, $\operatorname{pd}(S/L') = 3$ and $e(S/L') = 6$; hence $L' = (x^3, y^3) : I$. ☐

Lemma 4.5. *Let*

$$I = \big(x,y\big)^3 + \big(x,y\big)\big(ax+by\big) + \big(a(ax+by)+cxy+dy^2, b(ax+by) - cx^2 - dxy\big),$$

where $\operatorname{ht}(x,y,a,b) = 4$ *and* $\operatorname{ht}(x,y,a,b,c,d) \geq 5$. *Then* I *is* $\mathfrak{p}$*-primary,* $\operatorname{pd}(S/I) = 4$*, and* $e(S/I) = 3$*. Moreover,*

$$L = \big(x^3, y^3\big) : I = \big(y^3, x^3, x^2y^2, ax^2y - bxy^2, a^2x^2 - abxy + b^2y^2 + cx^2y - dxy^2\big)$$

and $\operatorname{pd}(S/L) = 3$.

Proof. Consider the complex

$$S \xleftarrow{\partial_1} S^8 \xleftarrow{\partial_2} S^{12} \xleftarrow{\partial_3} S^6 \xleftarrow{\partial_4} S \longleftarrow 0,$$

where

$$\partial_1 = \begin{pmatrix} a^2x + aby + cxy + dy^2 \\ abx - cx^2 + b^2y - dxy \\ ax^2 + bxy \\ x^3 \\ axy + by^2 \\ x^2y \\ xy^2 \\ y^3 \end{pmatrix}^{\mathsf{T}},$$

$$\partial_2 = \begin{pmatrix}
-b & -x & 0 & 0 & -y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a & 0 & -x & 0 & 0 & -y & 0 & 0 & 0 & 0 & 0 & 0 \\
c & a & b & -x & 0 & 0 & -y & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -c & a & 0 & 0 & 0 & 0 & -y & 0 & 0 & 0 \\
d & 0 & 0 & 0 & a & b & x & -x & 0 & -y & 0 & 0 \\
0 & c & -d & b & 0 & -c & 0 & a & x & 0 & -y & 0 \\
0 & d & 0 & 0 & c & -d & 0 & b & 0 & a & x & -y \\
0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & b & 0 & x
\end{pmatrix},$$

$$\partial_3 = \begin{pmatrix}
x & y & 0 & 0 & 0 & 0 \\
-b & 0 & y & 0 & 0 & 0 \\
a & 0 & 0 & y & 0 & 0 \\
c & 0 & 0 & 0 & y & 0 \\
0 & -b & -x & 0 & 0 & 0 \\
0 & a & 0 & -x & 0 & 0 \\
0 & c & a & b & -x & 0 \\
d & c & 0 & 0 & -x & y \\
0 & 0 & 0 & -c & a & 0 \\
0 & d & 0 & 0 & 0 & -x \\
0 & 0 & c & -d & b & a \\
0 & 0 & d & 0 & 0 & b
\end{pmatrix},$$

$$\partial_4 = \begin{pmatrix} -y \\ x \\ -b \\ a \\ c \\ d \end{pmatrix}.$$

It is straightforward to check that this is a complex. Now we consider the ideals of minors of the expected ranks. $I_1(\partial_1) = I$, which has height 2. $x^7, y^7 \in I_7(\partial_2)$, so $\mathrm{ht}(I_7(\partial_2)) \geq 2$. One can check that $x^5, y^5, a^5, b^5 \in I_5(\partial_3)$, and hence $\mathrm{ht}(I_5(\partial_3)) \geq 4$. It is clear that $I_1(\partial_4) = (x,y,a,b,c,d)$; hence, $\mathrm{ht}(I_1(\partial_4)) \geq 4$. Therefore, the above complex is exact and I is unmixed exactly when $\mathrm{ht}(x,y,a,b,c,d) \geq 5$.

Finally note that $L' := \big(y^3, x^3, x^2y^2, ax^2y - bxy^2, a^2x^2 - abxy + b^2y^2 + cx^2y - dxy^2\big) \subseteq (x^3,y^3) : I$. Since $\mathrm{ht}(x,y,a,b) = 4$ and $\mathrm{ht}(x,y,1,ad+bc) > 3$, Lemma 4.1 implies that L' is $\mathfrak{p}$-primary, $\mathrm{pd}(S/L') = 3$ and $e(S/L') = 6$. Hence $L' = (x^3,y^3) : I$. □

Lemma 4.6. *If*

$$I = \big(x^2, xy, y^4, ax + by^3\big),$$

where $\mathrm{ht}(x,y,a,b) \geq 4$*, then* I *is* $\mathfrak{p}$*-primary,* $\mathrm{pd}(S/I) \leq 3$*, and* $e(S/I) = 4$.

Proof. Consider the complex

$$S \xleftarrow{\begin{pmatrix} x^2 \\ xy \\ ax+by^3 \\ y^4 \end{pmatrix}^{\mathsf{T}}} S^4 \xleftarrow{\begin{pmatrix} -y & -a & 0 & 0 \\ x & -by^2 & -a & -y^3 \\ 0 & x & y & 0 \\ 0 & 0 & -b & x \end{pmatrix}} S^4 \xleftarrow{\begin{pmatrix} a \\ -y \\ x \\ b \end{pmatrix}} S \longleftarrow 0.$$

It is easy to check that this is complex is exact and resolves S/I and so $\mathrm{pd}(S/I) \leq 3$ and I is unmixed. Since $\sqrt{I} = \mathfrak{p}$ and since $1, x, y, y^2$ form a basis of $(S/I)_{\mathfrak{p}}$, I is $\mathfrak{p}$-primary and $e(S/I) = 4$. □

Lemma 4.7. *If*

$$I = \big(x^2, xy^2, y^3 + axy, bxy + c(y^2 + ax)\big),$$

where $\mathrm{ht}(x,y,b,c) = 4$*, then* I *is* $\mathfrak{p}$*-primary,* $\mathrm{pd}(S/I) = 3$*, and* $e(S/I) = 4$.

Proof. Consider the complex

$$S \xleftarrow{\begin{pmatrix} x^2 \\ acx + bxy + cy^2 \\ axy + y^3 \\ xy^2 \end{pmatrix}^{\mathsf{T}}} S^4 \xleftarrow{\begin{pmatrix} -ac-by & 0 & -ay & -y^2 \\ x & -y & 0 & 0 \\ 0 & c & x & 0 \\ -c & b & -y & x \end{pmatrix}} S^4 \xleftarrow{\begin{pmatrix} -y \\ -x \\ c \\ b \end{pmatrix}} S \longleftarrow 0.$$

It is easy to check that this complex is exact and resolves S/I, thus $\mathrm{pd}(S/I) = 3$ and I is unmixed. Since $\sqrt{I} = \mathfrak{p}$ and since $1, x, y, xy$ form a basis of $(S/I)_{\mathfrak{p}}$, I is $\mathfrak{p}$-primary and $e(S/I) = 4$.

□

Lemma 4.8. *If*

$$I = \big(x^2, xy^2, y^4, ay^2 + bx\big),$$

where $\operatorname{ht}(x, y, b, c) = 4$*, then* I *is* $\mathfrak{p}$*-primary,* $\operatorname{pd}(S/I) = 3$*, and* $e(S/I) = 4$*.*

Proof. The proof here follows that of [5, Lemma 10] exactly with $e = 2$ and y^2 replacing y. □

Lemma 4.9. *If*

$$I = \big(x^3, xy, y^3, ax^2 + by^2\big),$$

where $\operatorname{ht}(x, y, a, b) = 4$*, then* I *is* $\mathfrak{p}$*-primary,* $\operatorname{pd}(S/I) = 3$*, and* $e(S/I) = 4$*.*

Proof. Consider the complex

$$S \xleftarrow{\begin{pmatrix} xy \\ ax^2+by^2 \\ x^3 \\ y^3 \end{pmatrix}^{\mathsf{T}}} S^4 \xleftarrow{\begin{pmatrix} by & -ax & -x^2 & -y^2 \\ -x & y & 0 & 0 \\ a & 0 & y & 0 \\ 0 & -b & 0 & x \end{pmatrix}} S^4 \xleftarrow{\begin{pmatrix} -y \\ -x \\ a \\ -b \end{pmatrix}} S \longleftarrow 0.$$

Note that $I_1(\partial_1) = I$, $x^4, y^4 \in I_3(\partial_2)$, and $I_1(\partial_3) = (x, y, a, b)$. The rest follows as in the previous examples. □

Lemma 4.10. *If*

$$I = \big(x, y\big)^3 + \big(axy - by^2, ax^2 - cy^2, bx^2 - cxy\big),$$

where $\operatorname{ht}(x, y, a, b, c) = 5$*, then* I *is* $\mathfrak{p}$*-primary,* $\operatorname{pd}(S/I) = 4$*, and* $e(S/I) = 4$*.*

Proof. Consider the complex

$$S \xleftarrow{\partial_1} S^7 \xleftarrow{\partial_2} S^{10} \xleftarrow{\partial_3} S^5 \xleftarrow{\partial_4} S \longleftarrow 0,$$

where

$$\partial_1 = \begin{pmatrix} ax^2 - cy^2 \\ bx^2 - cxy \\ x^3 \\ axy - by^2 \\ x^2y \\ xy^2 \\ y^3 \end{pmatrix}^{\mathsf{T}},$$

$$\partial_2 = \begin{pmatrix} -b & -x & 0 & -y & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & -x & 0 & 0 & -y & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 & 0 & -y & 0 & 0 & 0 \\ c & 0 & 0 & x & -x & 0 & 0 & -y & 0 & 0 \\ 0 & 0 & -c & 0 & a & b & x & 0 & -y & 0 \\ 0 & -c & 0 & b & -b & -c & 0 & a & x & -y \\ 0 & 0 & 0 & -c & 0 & 0 & 0 & -b & 0 & x \end{pmatrix}^{\mathsf{T}},$$

$$\partial_3 = \begin{pmatrix} x & y & 0 & 0 & 0 \\ -b & 0 & y & 0 & 0 \\ a & 0 & 0 & y & 0 \\ 0 & -b & -x & 0 & 0 \\ c & -b & -x & 0 & y \\ 0 & a & 0 & -x & 0 \\ 0 & 0 & a & b & 0 \\ 0 & c & 0 & 0 & -x \\ 0 & 0 & 0 & -c & a \\ 0 & 0 & -c & 0 & -b \end{pmatrix},$$

$$\partial_4 = \begin{pmatrix} -y \\ x \\ -b \\ a \\ c \end{pmatrix}.$$

One checks that $I = I_1(\partial_1)$, $x^6, y^6 \in I_6(\partial_2)$, $x^4, y^4, a^4, b^4, c^4 \in I_4(\partial_3)$ and $I_1(\partial_4) = (x, y, a, b, c)$. It follows that this is a resolution of S/I and I is unmixed. Since $\sqrt{I} = \mathfrak{p}$ and $H_{(S/I)_\mathfrak{p}} = (1, 2, 1)$, we have that I is $\mathfrak{p}$-primary and $e(S/I) = 4$. □

Lemma 4.11. *If*

$$I = \big(x, y\big)^3 + \big(ax^2 + bxy + cy^2\big),$$

where $\operatorname{ht}(x, y, a, b, c) \geq 4$*. Then I is $\mathfrak{p}$-primary,* $\operatorname{pd}(S/I) = 3$*, and* $e(S/I) = 5$*.*

Proof. Consider the complex

$$S \xleftarrow{\begin{pmatrix} ax^2+bxy+cy^2 \\ x^3 \\ x^2y \\ xy^2 \\ y^3 \end{pmatrix}^{\intercal}} S^5 \xleftarrow{\begin{pmatrix} -x & -y & 0 & 0 & 0 \\ a & 0 & -y & 0 & 0 \\ b & a & x & -y & 0 \\ c & b & 0 & x & -y \\ 0 & c & 0 & 0 & x \end{pmatrix}} S^5 \xleftarrow{\begin{pmatrix} y \\ -x \\ a \\ b \\ c \end{pmatrix}} S \longleftarrow 0.$$

We compute that $I = I_1(\partial_1)$, $x^4, y^4 \in I_4(\partial_2)$ and $I_1(\partial_3) = (x, y, a, b, c)$. It follows that this is a resolution of S/I and I is unmixed. As $\sqrt{I} = \mathfrak{p}$ and $H_{(S/I)_\mathfrak{p}} = (1, 2, 2)$, it follows that I is $\mathfrak{p}$-primary and $e(S/I) = 5$. □

Lemma 4.12. *If*

$$I = \big(x, y\big)^3 + \big(ax^2 + bxy + cy^2, dx^2 + exy + fy^2\big),$$

where $\operatorname{ht}(x, y, ae - bd, af - cd, bf - ce) \geq 4$*, then I is $\mathfrak{p}$-primary,* $\operatorname{pd}(S/I) = 3$ *and* $e(S/I) = 4$*. Moreover,*

$$L = \big(x^3, y^3\big) : I = \big(x, y\big)^3 + \big((ae - bd)x^2 + (af - cd)xy + (bf - ce)y^2\big)$$

L is $\mathfrak{p}$-primary, $e(S/L) = 5$ *and* $\operatorname{pd}(S/L) = 3$*.*

Proof. Consider the complex

$$S \xleftarrow{\partial_1} S^6 \xleftarrow{\partial_2} S^7 \xleftarrow{\partial_3} S^2 \longleftarrow 0,$$

where

$$\partial_1 = \begin{pmatrix} ax^2+bxy+cy^2 \\ dx^2+exy+fy^2 \\ x^3 \\ x^2y \\ xy^2 \\ y^3 \end{pmatrix}^{\mathsf{T}},$$

$$\partial_2 = \begin{pmatrix} -x & 0 & -y & 0 & 0 & 0 & 0 \\ 0 & -x & 0 & -y & 0 & 0 & 0 \\ a & d & 0 & 0 & -y & 0 & 0 \\ b & e & a & d & x & -y & 0 \\ c & f & b & e & 0 & x & -y \\ 0 & 0 & c & f & 0 & 0 & x \end{pmatrix},$$

$$\partial_3 = \begin{pmatrix} y & 0 \\ 0 & y \\ -x & 0 \\ 0 & -x \\ a & d \\ b & e \\ c & f \end{pmatrix}.$$

One checks that $I = I_1(\partial_1)$, $x^5, y^5 \in I_2(\partial_2)$ and $x^2, y^2, ae-bd, af-cd, bf-de \in I_2(\partial_3)$. It follows that this is a resolution of S/I and I is unmixed. As $\sqrt{I} = \mathfrak{p}$ and $H_{(S/I)_\mathfrak{p}} = (1,2,1)$, we see that I is $\mathfrak{p}$-primary, and $e(S/I) = 4$.

It follows from Lemma 4.11 that L is $\mathfrak{p}$-primary, $\operatorname{pd}(S/L) = 3$ and $e(S/L) = 5$. Since the inclusion $L \subseteq (x^3, y^3) : I$ is clear and both ideals have the same height and multiplicity, they must be equal. □

Lemma 4.13. *If*

$$I = \big(x,y\big)^4 + \big(x,y\big)\big(ax+by\big) + \big(c(ax+by) + dx^3 + ex^2y + fxy^2 + gy^3\big),$$

where $\operatorname{ht}(x,y,a,b) = 4$ *and* $\operatorname{ht}(x,y,c,ga^3 - fa^2b + eab^2 - db^3) = 4$, *then* I *is* $\mathfrak{p}$*-primary,* $\operatorname{pd}(S/I) = 3$ *and* $e(S/I) = 4$.

Proof. Consider the (nonminimal) complex

$$S \xleftarrow{\partial_1} S^8 \xleftarrow{\partial_2} S^{11} \xleftarrow{\partial_3} S^4 \longleftarrow 0,$$

where

$$\partial_1 = \begin{pmatrix} ax^2+bxy \\ axy+by^2 \\ dx^3+ex^2y+fxy^2+gy^3+acx+bcy \\ x^4 \\ x^3y \\ x^2y^2 \\ xy^3 \\ y^4 \end{pmatrix}^{\mathsf{T}},$$

$$\partial_2 = \begin{pmatrix} -y & -x^2 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & -x^2 & c & 0 & -xy & 0 & -y^2 & 0 & 0 \\ 0 & 0 & -x & 0 & -y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & d & 0 & 0 & -y & 0 & 0 & 0 & 0 & 0 \\ 0 & b & e & a & d & x & 0 & -y & 0 & 0 & 0 \\ 0 & 0 & f & b & e & 0 & a & x & 0 & -y & 0 \\ 0 & 0 & g & 0 & f & 0 & b & 0 & a & x & -y \\ 0 & 0 & 0 & 0 & g & 0 & 0 & 0 & b & 0 & x \end{pmatrix},$$

$$\partial_3 = \begin{pmatrix} -x^2 & c & 0 & 0 \\ y & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ -x & 0 & y & 0 \\ 0 & -x & 0 & 0 \\ a & d & 0 & 0 \\ 0 & 0 & -x & y \\ b & e & a & 0 \\ 0 & 0 & 0 & -x \\ 0 & f & b & a \\ 0 & g & 0 & b \end{pmatrix}.$$

One checks that $I = I_1(\partial_1)$, $x^8, y^8 \in I_7(\partial_2)$ and $x^4, y^4, a^3c, b^3c, db^3 - eab^2 + fa^2b - ga^3 \in I_4(\partial_3)$. It follows that this is a resolution of S/I and I is unmixed. Since $\sqrt{I} = \mathfrak{p}$ and $H_{(S/I)_\mathfrak{p}} = (1, 1, 1, 1)$, I is $\mathfrak{p}$-primary and $e(S/I) = 4$. □

Note that the following lemma includes the case when $c, d, e \in k$ and $\operatorname{pd}(S/I) = 3$.

Lemma 4.14. *If*

$$I = (x, y)^4 + (x, y)(ax + by) + \big(a(ax + by) + cx^2y + dxy^2 + ey^3, b(ax + by) - cx^3 - dx^2y - exy^2\big),$$

where $\operatorname{ht}(x, y, a, b) = 4$ *and* $\operatorname{ht}(x, y, a, b, c, d, e) \geq 5$*, then* I *is* $\mathfrak{p}$*-primary,* $\operatorname{pd}(S/I) \leq 4$*, and* $e(S/I) = 4$*.*

Proof. Consider the complex

$$S \xleftarrow{\partial_1} S^9 \xleftarrow{\partial_2} S^{14} \xleftarrow{\partial_3} S^7 \xleftarrow{\partial_4} S \longleftarrow 0,$$

where

$$\partial_1 = \begin{pmatrix} ax^2+bxy \\ x^4 \\ axy+by^2 \\ x^3y \\ x^2y^2 \\ xy^3 \\ y^4 \\ a^2x+aby+cx^2y+dxy^2+ey^3 \\ abx-cx^3+b^2y-dx^2y-exy^2 \end{pmatrix}^{\mathsf{T}},$$

$$\partial_2 = \begin{pmatrix} -y & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -y & 0 & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -y & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -y & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -y & x & 0 & 0 \\ -a & 0 & 0 & -c & -d & -e & 0 & x & 0 \\ -b & c & 0 & d & e & 0 & 0 & 0 & x \\ -x^2 & a & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 & -c & -d & -e & y & 0 \\ 0 & 0 & -b & c & d & e & 0 & 0 & y \\ 0 & 0 & -x^2 & a & b & 0 & 0 & 0 & 0 \\ 0 & 0 & -xy & 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & -y^2 & 0 & 0 & a & b & 0 & 0 \\ cx+dy & 0 & ey & 0 & 0 & 0 & 0 & -b & a \end{pmatrix}^{\mathsf{T}},$$

$$\partial_3 = \begin{pmatrix} a & b & x^2 & 0 & 0 & dx & cx+dy \\ 0 & -c & -a & 0 & 0 & 0 & 0 \\ c & -d & -b & -a & 0 & 0 & 0 \\ d & -e & 0 & -b & -a & 0 & 0 \\ e & 0 & 0 & 0 & -b & 0 & 0 \\ -y & 0 & 0 & 0 & 0 & b & 0 \\ 0 & -y & 0 & 0 & 0 & -a & 0 \\ 0 & 0 & -y & 0 & 0 & c & 0 \\ x & 0 & 0 & 0 & 0 & 0 & b \\ 0 & x & 0 & 0 & 0 & 0 & -a \\ 0 & 0 & x & -y & 0 & d & c \\ 0 & 0 & 0 & x & -y & e & d \\ 0 & 0 & 0 & 0 & x & 0 & e \\ 0 & 0 & 0 & 0 & 0 & x & y \end{pmatrix},$$

$$\partial_4 = \begin{pmatrix} -b \\ a \\ -c \\ -d \\ -e \\ -y \\ x \end{pmatrix}.$$

One checks that $I = I_1(\partial_1)$, $x^9, y^9 \in I_8(\partial_2)$, $x^7, y^7, a^6, b^6 \in I_6(\partial_3)$ and $I_1(\partial_4) = (x, y, a, b, c, d, e)$. It follows that this is a resolution of S/I and I is unmixed. Since $\sqrt{I} = \mathfrak{p}$ and the Hilbert function of $(S/I)_{\mathfrak{p}}$ is $(1, 2, 1)$, we have that I is $\mathfrak{p}$-primary and $e(S/I) = 4$. □

Lemma 4.15. *If*

$$I = \big(x, y\big)^4 + \big(x, y\big)\big(ax + by\big) + \big(b(ax + by) + ay^2, x(ax + by) - y^3\big),$$

where $\operatorname{ht}(x, y, a, b) = 4$*, then* I *is* $\mathfrak{p}$*-primary,* $\operatorname{pd}(S/I) = 3$*, and* $e(S/I) = 4$*.*

Proof. Consider the complex

$$S \xleftarrow{\partial_1} S^7 \xleftarrow{\partial_2} S^{10} \xleftarrow{\partial_3} S^4 \longleftarrow 0,$$

where

$$\partial_1 = \begin{pmatrix} abx + b^2y + ay^2 & ax^2 + bxy - y^3 & x^4 & x^3y & x^2y^2 & xy^3 & y^4 \end{pmatrix},$$

$$\partial_2 = \begin{pmatrix} 0 & 0 & 0 & bx - y^2 & x^2 & -x^2 & 0 & -xy & 0 & 0 \\ -x^2 & -xy & 0 & -b^2 & -bx - y^2 & bx & 0 & by & 0 & 0 \\ a & 0 & -y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & a & x & 0 & 0 & 0 & -y & 0 & 0 & 0 \\ -y & b & 0 & 0 & 0 & a & x & 0 & -y & 0 \\ 0 & -y & 0 & 0 & 0 & b & 0 & a & x & -y \\ 0 & 0 & 0 & a & -y & 0 & 0 & b & 0 & x \end{pmatrix},$$

$$\partial_3 = \begin{pmatrix} y & 0 & 0 & 0 \\ -x & 0 & y & 0 \\ a & 0 & 0 & 0 \\ 0 & -x & 0 & 0 \\ 0 & b & -x & 0 \\ 0 & 0 & -x & y \\ b & 0 & a & 0 \\ 0 & y & 0 & -x \\ -y & 0 & b & a \\ 0 & a & -y & b \end{pmatrix}.$$

One checks that $I = I_1(\partial_1)$, $x^8, y^8 \in I_6(\partial_2)$, $x^4, y^4, a^4, b^4 \in I_4(\partial_3)$. It follows that this is a resolution of S/I and I is unmixed. Since $\sqrt{I} = \mathfrak{p}$ and the Hilbert function of $(S/I)_{\mathfrak{p}}$ is $(1, 1, 1, 1)$, we have that I is $\mathfrak{p}$-primary and $e(S/I) = 4$. □

5. Application to the Projective Dimension of Three Cubics

For this section, S denotes a polynomial ring over any field k and $I = (f, g, h)$ an ideal generated by 3 cubic forms with $\mathrm{ht}(I) = 2$. After possibly taking a linear combination of the generators we may further assume that $\mathrm{ht}(f, g) = \mathrm{ht}(f, h) = 2$. Let I^{un} denote the unmixed part of I, that is, the intersection of all height 2 primary components of I. We wish to show that the results of the previous sections allow us to conclude that $\mathrm{pd}(S/I) \leq 5$ if I^{un} or $(f, g) : I$ have the form of one of the ideals in Theorems 2.1 or 3.1. By extending the base field we may also assume that k is algebraically closed.

Two ideals J and K are said to be *(directly) linked* via a complete intersection C if $J = C : L$ and $L = C : J$. Note that since (f, g) is a complete intersection, $(f, g) : I = (f, g) : h = (f, g) : I^{un}$ is an unmixed ideal directly linked to I^{un}. We also make use of the following facts:

Proposition 5.1. *Let $I = (f, g, h)$ be as above and J and L be homogeneous unmixed ideals. Then*

(1) $\mathrm{pd}(S/I) \leq \mathrm{pd}(S/((f, g) : I)) + 1$. [5, Theorem 7]
(2) *If all cubics in I^{un} can be expressed in terms of a regular sequence of length r, then* $\mathrm{pd}(S/I) \leq r$. [1, Lemma 3.3]
(3) *If I^{un} contains a quadric, then* $\mathrm{pd}(S/I) \leq 4$. [5, Theorem 16]
(4) *If $(f, g) : I$ contains a quadric and $e(S/I) \leq 5$, then* $\mathrm{pd}(S/I) \leq 4$. [5, Theorem 17]
(5) *If J and L are directly linked, then J is Cohen-Macaulay if and only if L is Cohen-Macaulay.* [17] *See* [11, Proposition 2.5].
(6) *If L and L' are directly linked to the same unmixed ideal J, then* $\mathrm{pd}(S/L) = \mathrm{pd}(S/L')$. [6, Lemma 2.6]
(7) *If L and L' are directly linked to an unmixed ideal J by complete intersections $\underline{c} = c_1, \ldots, c_h$ and $\underline{c'} = c'_1, \ldots, c'_h$, where $\deg(c_i) = \deg(c'_i)$ for all i, then S/L and S/L' have identical Hilbert functions. In particular, the minimal degree of a minimal generator and the number of generators of minimal degree are the same for L and L'.*

Proof. References for (1) - (6) are listed above.

For part (7), note that we have a short exact sequence

$$0 \to ((\underline{c}) : J)/(\underline{c}) \cong L/(\underline{c}) \to S/(\underline{c}) \to S/L \to 0.$$

By [9, Lemma 3.1], $L/(\underline{c}) \cong \mathrm{Ext}^2_S(S/L, S)(-d)$, where $d = \sum_{i=1}^{h} c_i$. (The grading follows by tracking the grading through the proof of [2, Lemma 1.2.4].) Let $\mathbf{F}_\bullet$ denote the minimal graded free resolution of $\mathrm{Ext}^2_S(S/L, S)(-d)$, let $\mathbf{K}(\underline{c})$ denote the Koszul complex on $c_1, \ldots, c_h$, which minimally resolves $S/(\underline{c})$. Let $\varphi : \mathbf{F} \to \mathbf{K}(\underline{c})$ denote the map of complexes representing a lift of the first map above. Then the mapping cone $cone(\varphi)$ is a (possibly non-minimal) graded free resolution of S/L whose graded Betti numbers depend only on J and the degrees of $c_1, \ldots, c_h$. Since the Hilbert function of an ideal is determined by any graded free resolution, the claim follows. □

In part (7) of the previous proposition, we cannot make the stronger claim that number and degrees of *all* the minimal generators of L and L' are the same. Consider the following example: Set $S = k[x, y]$ and let $J = (x, y^2)$. In this case:

$$L = (x^3, y^2) : J = (x^2, y^2)$$
$$L' = (y^3, x^2) : J = (x^2, xy, y^3).$$

Both L and L' are generated in degree at least 2 and have 2 minimal generators that are quadrics; however the entire generating sets of each differ in size and degrees. This occurs because the mapping cone $cone(\varphi)$ from the above proof is minimal in the case of L' but is not minimal in the case of L.

For the rest of the paper, let x, y be two linearly independent linear forms in S and $\mathfrak{p} = (x, y)$.

Proposition 5.2. *Let* $I = (f, g, h)$ *as above. Suppose* I^{un} *is* $\mathfrak{p}$*-primary with* $e(S/I^{un}) = 3$*. Then* $\operatorname{pd}(S/I) \leq 4$*.*

Proof. I^{un} is of one of the seven forms in Theorem 2.1. In cases (i) and *(ii)*, I^{un} is Cohen-Macaulay, thus Proposition 5.1(5) yields $\operatorname{pd}(S/((f, g) : I)) = 2$, and $\operatorname{pd}(S/I) \leq 3$ by Proposition 5.1(1).

In cases *(iii)* to *(vii)*, we see from Lemmas 4.2, 4.3, 4.4 and 4.5 that I^{un} is directly linked to an ideal L with $\operatorname{pd}(S/L) \leq 3$. Hence $\operatorname{pd}(S/((f, g) : I)) \leq 3$ by Proposition 5.1(6) and $\operatorname{pd}(S/I) \leq 4$ by Proposition 5.1(1).

In case *(viii)*, all three cubics are expressible in terms of the regular sequence x, y, a, b. Hence $\operatorname{pd}(S/I) \leq 4$ by Proposition 5.1(2). □

Proposition 5.3. *Let* $I = (f, g, h)$ *as above. Suppose* $L = (f, g) : I$ *is* $\mathfrak{p}$*-primary with* $e(S/L) = 3$*. Then* $\operatorname{pd}(S/I) \leq 5$*.*

Proof. In cases *(i)*$-$*(vii)*, we note that the projective dimension of $S/((f, g) : I)$ is at most 4 by Theorem 2.1. Hence $\operatorname{pd}(S/I) \leq 5$ by Proposition 5.1(1).

In case *(viii)*, we have that $J' := \big(x, y\big)^3 + \big(x, y\big)\big(ax + by\big) \subseteq (f, g) : I = (f, g) : I^{un}$. Set $L' = (x^3, y^3) : J'$. By Lemma 4.12 $L' = \big(x, y\big)^3 + \big(a^2x^2 - abxy + b^2y^2\big)$. In particular, L' contains no linear or quadric forms. By Proposition 5.1(7), $H' = (f, g) : J'$ is also generated in degree at least 3. Since the minimal generators of (f, g) and J' lie in $k[x, y, a, b]$, so do those of H'. It follows that all cubics in $I \subseteq I^{un} \subseteq H'$ are expressible in terms of the regular sequence x, y, a, b, and so $\operatorname{pd}(S/I) \leq 4$ by Proposition 5.1(2).

□

Proposition 5.4. *Let* $I = (f, g, h)$ *as above. Suppose* I^{un} *is* $\mathfrak{p}$*-primary and* $e(S/I^{un}) = 4$*. Then* $\operatorname{pd}(S/I) \leq 5$*.*

Proof. The ideal I^{un} has one of the forms in Theorem 3.1. In cases *(i)* through *(x)*, *(xvii)*, and *(xviii)*, I^{un} contains a quadric. Thus $\operatorname{pd}(S/I) \leq 4$ by Proposition 5.1(3).

I^{un} cannot have the form of cases *(xi)*, *(xix)*, *(xx)* or *(xxiii)*, because I^{un} contains a regular sequence of cubics of height 2, while in these cases the cubics form an ideal of height at most 1.

In all remaining cases, the cubics in I^{un} are expressible in terms of a regular sequence of at most 5 forms and so $\mathrm{pd}(S/I) \leq 5$ by Proposition 5.1(2). □

Proposition 5.5. *Let* $I = (f, g, h)$ *as above. Suppose* $L = (f, g) : I$ *is* $\mathfrak{p}$*-primary and* $e(S/L) = 4$*. Then* $\mathrm{pd}(S/I) \leq 5$*.*

Proof. The ideal L has one of the forms in Theorem 3.1. In cases *(i)* through *(x)* as well as *(xvii)* and *(xviii)*, L contains a quadric. Thus $\mathrm{pd}(S/I) \leq 4$ by Proposition 5.1(4).

In cases *(xi)*, *(xix)*, *(xx)* and *(xxiii)*, the cubics in L are contained in an ideal of height at most one contradicting that $f, g \in L$.

In cases *(xii)* through *(xvi)*, $\mathrm{pd}(S/L) \leq 4$ by Lemmas 4.12, 4.10, 4.13, 4.14 and 4.15, respectively. By Proposition 5.1(1), we obtain $\mathrm{pd}(S/I) \leq 5$.

In case *(xxii)*, we have $J = \big(x, y\big)^3 + \big(ax^2 + bxy + cy^2\big) \subset L$. We may assume $a, b, c, \in S_1$ or else we are in case *(xxi)*. Since $f, g \in J$ are cubics and all cubics in J lie in $k[x, y, a, b, c]$, then $f, g \in R = k[x, y, a, b, c]$. Then also $H = (f, g) : J$ is extended from an ideal in R; in particular there exists a minimal generating set of H written purely in terms of x, y, a, b, c. By Lemma 4.11, $\big(x^3, y^3\big) : J$ is generated by 7 cubic forms; by Proposition 5.1(7) H is generated by 7 cubic forms (and potentially higher degree forms) all lying in $R = k[x, y, a, b, c]$. Since $I^{un} = (f, g) : L \subseteq (f, g) : J = H$ it follows that all cubics in I are expressible in terms of the linear forms x, y, a, b, c and so $\mathrm{pd}(S/I) \leq 5$ by Proposition 5.1(2).

Finally, in case *(xxi)*, f, g lie in $R = k[x, y]$, in particular $(f, g)R$ is $(x, y)R$-primary and generated by 2 cubics. Since $(x^3, y^3)R : (x, y)^3R = (x, y)^2R$, then by Proposition 5.1(7) also $(f, g)R : (x, y)^3R = (x, y)^2R$ (because $(f, g)R : (x, y)^3R$ contains three minimal generators of degree 2 in $R = k[x, y]$ and no generators in strictly smaller degree). Extending to the larger polynomial ring S we obtain (by flatness)

$$I^{un} = (f, g) : L \subseteq (f, g) : (x, y)^3 = [(f, g)R : (x, y)^3R]S = (x, y)^2S.$$

Since $h \in I^{un} = (x, y)^2$ there are linear forms a, b, c such that $h = ax^2 + bxy + cy^2$. Since $f, g \in R = k[x, y]$, then f, g, h all lie in $k[x, y, a, b, c]$, thus $\mathrm{pd}(S/I) \leq 5$ by Proposition 5.1(2). This completes the proof. □

Acknowledgements

The authors thank the referee for finding several errors and helping to greatly improve this paper.

References

[1] T. Ananyan and M. Hochster, Ideals generated by quadratic polynomials, Math. Res. Lett. **19** (2012), 233-244.

[2] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Stud. Adv. Math., vol. 39, Cambridge University Press, Cambridge, 1993.

[3] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry*, Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995.

[4] D. Eisenbud and S. Goto: *Linear free resolutions and minimal multiplicity*, J. Algebra **88** (1984), 89–133.

[5] B. Engheta, *On the projective dimension and the unmixed part of three cubics*, J. Algebra **316** (2007), 715–734.

[6] B. Engheta, *Bounds on projective dimension*, Ph.D. thesis, University of Kansas, 2005.

[7] B. Engheta, *A bound on the projective dimension of three cubics*, J. Sym. Comp. **45** (2010), 60–73.

[8] G. Fløystad, J. McCullough, and I. Peeva, *Three themes of syzygies*, Bull. Amer. Math. Soc. (N.S.) **53** (2016), no. 3, 415–435.

[9] C. Huneke, P. Mantero, J. McCullough and A. Seceleanu, *Multiple structures on linear varieties with high projective dimension*, J. Algebra **447** (2016), 183–205.

[10] C. Huneke, P. Mantero, J. McCullough and A. Seceleanu, A Tight Bound on the Projective Dimension of 4 Quadrics, arXiv:1403.6334 (submitted).

[11] C. Huneke and B. Ulrich, *The structure of linkage*, Ann. of Math. **126** (1987), 277–334.

[12] N. Manolache, *Codimension two linear varieties with nilpotent structures*, Math. Zeitschrift **210** (1992), no. 4, 573–580.

[13] N. Manolache, *Cohen-Macaulay nilpotent schemes*, in: Recent Advances in Geometry and Topology, Cluj Univ. Press, Cluj-Napoca (2004), pp. 235-248.

[14] Macaulay 2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/

[15] J. McCullough and A. Seceleanu, *Bounding projective dimension*, Commutative Algebra, Springer-Verlag London Ltd., London, 2012.

[16] I. Peeva and M. Stillman, Open problems on syzygies and Hilbert functions, J. Commut. Algebra **1** (2009), no. 1, 159–195.

[17] C. Peskine and L. Szpiro, Liaison des variétés algebriques, Invent. Math. **26** (1974), 271–302.

[18] I. Swanson and C. Huneke, *Integral closure of ideals, rings, and modules*, London Mathematical Society Lecture Note Series, 336. Cambridge University Press, Cambridge, 2006.

[19] J. Vatne, *Multiple structures and Hartshorne's conjecture*, Comm. Algebra, **37** (2009), no. 11, 3861-3873.

University of Arkansas, Department of Mathematics, Fayetteville, AR 72701

E-mail address: pmantero@uark.edu

Rider University, Department of Mathematics, Lawrenceville, NJ 08648

E-mail address: jmccullough@rider.edu