1. Show that the determinant of
$$\begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 \end{pmatrix}$$
is nonzero.

   Proof. If the above matrix is $n \times n$ then
$$\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix} \begin{pmatrix} n-2 & -1 & -1 & \cdots & -1 \\ -1 & n-2 & -1 & \cdots & -1 \\ -1 & -1 & n-2 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & -1 & -1 & \cdots & n-2 \end{pmatrix} = \begin{pmatrix} -(n-1) & 0 & 0 & \cdots & 0 \\ 0 & -(n-1) & 0 & \cdots & 0 \\ 0 & 0 & -(n-1) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -(n-1) \end{pmatrix}$$
   Clearly the determinant of the right hand matrix is just the product of the diagonal entries, which is nonzero (n is assumed to be at least 3). Since for any square matrices A and B, $\det(AB) = \det(A)\det(B)$, we must have that the determinant of both matrices on the left hand side are nonzero as well. (Check the $n = 2$ case by hand.) □

2. If $a, b, c > 0$, is it possible that each of the polynomials $P(x) = ax^2 + bx + c$, $Q(x) = cx^2 + ax + b$, $R(x) = bx^2 + cx + a$ has two real roots?

   Proof. No. Using the discriminant, a polynomial $Ax^2 + Bx + C$ has two real roots if and only if $B^2 - 4AC > 0$. If all three of the above polynomials have 2 real roots, we have that $b^2 > 4ac$ and $a^2 > 4bc$ and $c^2 > 4ab$. Multiplying we have $a^2b^2c^2 > 64a^2b^2c^2$, which is a contradiction. □

3. Consider a set S and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

   2001 A1. The hypothesis implies $((b * a) * b) * (b * a) = b$ for all $a, b \in S$ (by replacing a by $b * a$), and hence $a * (b * a) = b$ for all $a, b \in S$ (using $(b * a) * b = a$). □

4. In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

2002 A4. (partly due to David Savitt) Player 0 wins with optimal play. In fact, we prove that Player 1 cannot prevent Player 0 from creating a row of all zeroes, a column of all zeroes, or a 2×2 submatrix of all zeroes. Each of these forces the determinant of the matrix to be zero.

For $i, j = 1, 2, 3$, let A_{ij} denote the position in row i and column j. Without loss of generality, we may assume that Player 1's first move is at A_{11}. Player 0 then plays at A_{22}:

$$\begin{pmatrix} 1 & * & * \\ * & 0 & * \\ * & * & * \end{pmatrix}$$

After Player 1's second move, at least one of A_{23} and A_{32} remains vacant. Without loss of generality, assume A_{23} remains vacant; Player 0 then plays there.

After Player 1's third move, Player 0 wins by playing at A_{21} if that position is unoccupied. So assume instead that Player 1 has played there. Thus of Player 1's three moves so far, two are at A_{11} and A_{21}. Hence for i equal to one of 1 or 3, and for j equal to one of 2 or 3, the following are both true:

(a) The 2×2 submatrix formed by rows 2 and i and by columns 2 and 3 contains two zeroes and two empty positions.

(b) Column j contains one zero and two empty positions.

Player 0 next plays at A_{ij}. To prevent a zero column, Player 1 must play in column j, upon which Player 0 completes the 2×2 submatrix in (a) for the win.

Note: one can also solve this problem directly by making a tree of possible play sequences. This tree can be considerably collapsed using symmetries: the symmetry between rows and columns, the invariance of the outcome under reordering of rows or columns, and the fact that the scenario after a sequence of moves does not depend on the order of the moves (sometimes called "transposition invariance").

Note (due to Paul Cheng): one can reduce Determinant Tic-Tac-Toe to a variant of ordinary tic-tac-toe. Namely, consider a tic-tac-toe grid labeled as follows:

A_{11}	A_{22}	A_{33}
A_{23}	A_{31}	A_{12}
A_{32}	A_{13}	A_{21}

Then each term in the expansion of the determinant occurs in a row or column of the grid. Suppose Player 1 first plays in the top left. Player 0 wins by playing first in the top row, and second in the left column. Then there are only one row and column left for Player 1 to threaten, and Player 1 cannot already threaten both on the third move, so Player 0 has time to block both.

□

5. Two players, A and B, play the following game. Player A thinks of a polynomial with nonnegative integer coefficients. Player B can pick any value x and ask Player A for the value of the polynomial evaluated at x. Player B can pick any other value y and ask Player A for the value of the polynomial evaluated at y. Show that Player B can always determine all of the coefficients of Player A's polynomial.

Proof. Player B wins by doing the following. On her first guess, Player B asks for $f(1)$ and thus receives the sum of coefficients of f. Most importantly, this is an upper bound on any one coefficient. We write $n = f(1)$. Now Player B merely asks for $f(m)$ for any $m > n$. Since $m > n$, if we write $f(n+1)$ in base-n notation, we get that $f(x) = a_0 + a_1 x + \cdots + a_d x^d$ where $f(n+1) = a_0 + a_1 n^1 + a_2 n^2 + \cdots a_d n^d + \cdots$ and $a_0 + \cdots a_d = n$. □

6. Does there exist a polynomial $f(x)$ for which $xf(x-1) = (x+1)f(x)$?

Proof. No. Otherwise, for any positive integer n we would have

$$f(n) = \frac{nf(n-1)}{n+1} = \frac{(n-1)f(n-2)}{n+1} = \cdots = \frac{(-1)f(0)}{(n+1)} = \frac{(-1)0f(-1)}{n+1} = 0.$$

Hence every positive integer n is a root of f, a contradiction. $\square$