Putnam Practice Set #4

1. Show that the determinant of

$$\begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 \end{pmatrix}$$

   is nonzero.

2. If $a, b, c > 0$, is it possible that each of the polynomials $P(x) = ax^2 + bx + c$, $Q(x) = cx^2 + ax + b$, $R(x) = bx^2 + cx + a$ has two real roots?

3. Consider a set S and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

4. In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

5. Two players, A and B, play the following game. Player A thinks of a polynomial with nonnegative integer coefficients. Player B can pick any value x and ask Player A for the value of the polynomial evaluated at x. Player B can pick any other value y and ask Player A for the value of the polynomial evaluated at y. Show that Player B can always determine all of the coefficients of Player A's polynomial.

6. Does there exist a polynomial $f(x)$ for which $xf(x-1) = (x+1)f(x)$?