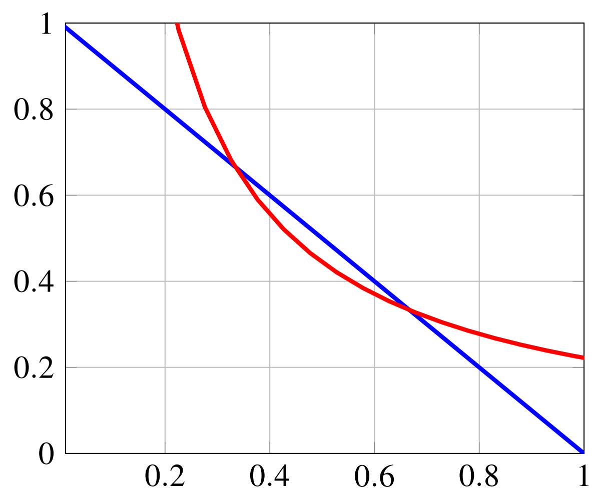
ISU Putnam Practice Set 3

Wednesday, February 10, 2021

1. What is the probability that the sum of two randomly chosen numbers in the interval $[0,1]$ does not exceed 1 and their product does not exceed $\frac{2}{9}$?

   Solution: Randomly picking two numbers in $[0,1]$ is equivalent to picking a point (x,y) randomly in the square $[0,1]\times[0,1]$, which has area one. To satisfy the two constraints, we must have $x+y\leq 1$ and $xy\leq\frac{2}{9}$. The functions $y=1-x$ and $y=\frac{2}{9x}$ intersect at $x=\frac{1}{3}$ and $x=\frac{2}{3}$. We compute the area between the curves as:

$$\int_{1/3}^{2/3}(1-x-\frac{2}{9x})dx=\left(x-\frac{x^2}{2}-\frac{2}{9}\ln x\right)\Big|_{1/3}^{2/3}=\frac{1}{6}-\frac{2}{9}\ln 2.$$



   The probability corresponds to the area of the triangle, which is $\frac{1}{2}$, minus the area we just computed, giving us a probability of $\frac{1}{3}+\frac{2}{9}\ln 2$.

2. If a needle of length 1 is dropped at random on a surface ruled with parallel lines at distance 2 apart, what is the probability that the needle will cross one of the lines?

   Solution: Mark the end fo the needle. Translations parallel to the given (horizontal) lines can be ignored; thus we can assume that the marked endpoint of the needle always falls on the same vertical. It's position is determined by the variables (x,θ), where x is the distance to the line right above and θ is the angle made with the horizontal.

The pair (x,θ) is randomly chosen from the region $[0,2)\times[0,2\pi)$. The area of this region is 4π. The probability that the needle will cross the upper horizontal line is

$$\frac{1}{4\pi}\int_0^{\pi} sin(\theta)d\theta = \frac{1}{2\pi}.$$

which is also the probability that the needle will cross the lower horizontal line. The probability for the needle to cross either the upper or lower horizontal line is therefore $\frac{1}{\pi}$.

3. What is the probability that three randomly chosen points on a circle form an acute triangle?

   Solution: Because of rotational symmetry, we can assume one of the points is fixed. Cut the circle at that point to create a segment. IN this new framework, the problem asks us to find the probability that two randomly chosen points on a segment cut it in three parts, none of which is larger than half of the original segment.
   Identify the segment with the interval $[0,1]$ and let the coordinates of the points be x and y. Then the possible choices can be identified with the interior of the square $[0,1]\times[0,1]$. The area of the total region is 1. The favorable region is:

   $$\left\{(x,y)|0<x<\frac{1}{2},\frac{1}{2}<y<\frac{1}{2}+x\right\}\cup\left\{(x,y)|\frac{1}{2}<x<1,x-\frac{1}{2}<y<\frac{1}{2}\right\}.$$

   The area of this region is $\frac{1}{4}$, which is also the probability.

4. Prove that $\frac{\gcd(m,n)}{n}\binom{n}{m}$ is an integer for all pairs of integers $n\geq m\geq 1$.

   Putnam 2000 B2. Solution: Let $a=\frac{m}{\gcd(m,n)}$ and $b=\frac{n}{\gcd(m,n)}$. Then

   $$\frac{a}{b}\binom{n}{m}=\frac{m}{n}\binom{n}{m}=\binom{n-1}{m-1}$$

   is an integer, so $b|a\binom{n}{m}$. But $\gcd(a,b)=1$, so $b|\binom{n}{m}$. Hence

   $$\frac{\gcd(m,n)}{n}\binom{n}{m}=\frac{1}{b}\binom{n}{m}$$

   is an integer.

5. Let f be a twice differentiable real-valued function satisfying

   $$f(x)+f''(x)=-xg(x)f'(x)$$

   where $g(x)\geq 0$ for all real x. Prove that $|f(x)|$ is bounded.

Putnam 1997 B2. Solution: Multiply both side of the equation by $2f'(x)$. We get

$$2f(x)f'(x)+2f'(x)f''(x)=-2xg(x)f'(x)^2.$$

The left side is the derivative of $f(x)^2+f'(x)^2$. The right side is non-negative for $x<0$ and non-positive for $x>0$. Thus $f(x)^2+f'(x)^2$ increases to its maximum value at $x=0$ and decreases thereafter. So it is bounded from above, and from below it is bounded by 0. Since $f(x)^2$, $f'(x)^2$ are non-negative, this implies that both of them are bounded too, which in turn implies that $f(x)$ and $f'(x)$ are bounded.