1. Show that
$$\sum_{n\in A}\frac{1}{n}<\infty,$$
where A is the set of positive integers that do not contain a '9' in their decimal expansion.

Proof. Let
$$S_n=\sum_{m\in A \text{ with } 10^{n-1}\le m<10^n}\frac{1}{m}.$$
Since the terms in the sum are positive, the sum is the same as $\sum_{n\to\infty} S_n$. Now we count the number of terms summed up in S_n. These are the number of natual numbers between 10^{n-1} and 10^n with no '9' in their decimal expansion. There are exactly n digits in any such natural number. Since there are no '9's and since the first digit cannot be 0, there are $8\times 9^{n-1}$ possible choices. For each of these choices, since $m\ge 10^{n-1}$, we have $\frac{1}{m}\le\frac{1}{10^{n-1}}$. Therefore
$$S_n\le\frac{8\times 9^{n-1}}{10^{n-1}}=8(\frac{9}{10})^{n-1}.$$
Since $\frac{9}{10}<1$, the series $\sum_{n=1}^{\infty}8(\frac{9}{10})^{n-1}$ converges as a geometric series. By comparison, the sum $\sum_{n=1}^{\infty}S_n$ also converges. □

2. Evaluate
$$\sum_{n=0}^{\infty}\text{Arccot}(n^2+n+1),$$
where Arccot (t) for $t\ge 0$ denotes the number θ in the interval $0<\theta\le\pi/2$ with $\cot\theta=t$.

A3 1986. There aren't many series that are easy to evaluate. Typically, to evaluate a series it is geometric, telescoping, or has a nice relation to a known Taylor series. The 1st and 3rd option don't seem hopeful, but maybe this is a telescoping series in disguise. Recall the difference formulae for sin and cos
$$\sin(A-B)=\sin(A)\cos(B)-\cos(A)\sin(B)$$
$$\cos(A-B)=\cos(A)\cos(B)+\sin(A)\sin(B).$$
Thus
$$\cot(A-B)=\frac{\cos(A-B)}{\sin(A-B)}=\frac{\cos(A)\cos(B)+\sin(A)\sin(B)}{\sin(A)\cos(B)-\cos(A)\sin(B)}=\frac{\cot(A)\cot(B)+1}{\cot(B)-\cot(A)}.$$
Setting $A=\cot^{-1}(a)$ and $B=\cot^{-1}(b)$ gives us
$$\cot^{-1}(a)-\cot^{-1}(b)=\cot^{-1}(\frac{ab+1}{b-a}).$$
Setting $b=n+1$ and $a=n$ gives us
$$\cot^{-1}(n)-\cot^{-1}(n+1)=\cot^{-1}(n^2+n+1).$$
Since $\cot^{-1}(n)\to 0$ as $n\to\infty$, our sum is telescoping and the sum is just $\cot^{-1}(0)=\frac{\pi}{2}$. □

3. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a non-zero function g defined on (a, b) such that the wrong product rule is true for x in (a, b).

Proof. Let $f(x) = e^{x^2}$. The equation $(fg)' = f'g'$ becomes

$$e^{x^2} 2xg(x) + e^{x^2} g'(x) = e^{x^2} 2xg'(x).$$

Dividing by e^{x^2} gives

$$2xg(x) = (2x - 1)g'(x).$$

Or just

$$1 + \frac{1}{2x-1} = \frac{g'(x)}{g(x)}.$$

Integrating both sides with respect to x gives

$$x + \frac{\ln|2x-1|}{2} = \ln|g(x)| + C,$$

for some constant C. Hence we can take

$$g(x) = e^x\sqrt{2x-1},$$

and consider the interval $(\frac{1}{2}, \infty)$, where $2x - 1 > 0$. Then

$$g'(x) = e^x\sqrt{2x-1} + e^x\frac{1}{\sqrt{2x-1}} = e^x\sqrt{2x-1}(1 + \frac{1}{2x-1}).$$

Therefore

$$(fg)' = \frac{d}{dx}\left(e^{x^2+1}\sqrt{2x-1}\right) = e^{x^2+x}\frac{1}{\sqrt{2x-1}} + e^{x^2+x}(2x+1)\sqrt{2x-1} = e^{x^2+x}\sqrt{2x-1}(2x+1+\frac{1}{2x-1})$$

and

$$f'g' = e^{x^2}2xe^x\sqrt{2x-1}(1 + \frac{1}{2x-1}) = e^{x^2+x}\sqrt{2x-1}(2x + 1 + \frac{1}{2x-1}).$$

□

4. Find all real-valued continuously differentiable functions f on the real line such that for all x,

$$(f(x))^2 = \int_0^x [(f(t))^2 + (f'(t))^2]\,dt + 1990.$$

Solution [B-1 1990]
Putting $y = f(x)$ and differentiating the relationship gives $2yy' = y^2 + (y')^2$ or $(y - y')^2 = 0$. So $y = y'$. Integrating, $y = Ae^x$. But $y(0) = \pm\sqrt{1990}$, so $f(x) = \pm\sqrt{1990}e^x$. The function is continuous, so we cannot "mix" the two solutions: either $f(x) = \sqrt{1990}e^x$ for all x, or $f(x) = -\sqrt{1990}e^x$ for all x.

5. Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$

Solution [A-3 (1998)]
If at least one of $f(a)$, $f'(a)$, $f''(a)$, or $f'''(a)$ vanishes at some point a, then we are done. Hence we may assume each of $f(x)$, $f'(x)$, $f''(x)$, and $f'''(x)$ is either strictly positive or strictly negative on the real line. By replacing $f(x)$ by $-f(x)$ if necessary, we may assume $f''(x) > 0$; by replacing $f(x)$ by $f(-x)$ if necessary, we may assume $f'''(x) > 0$. (Notice that these substitutions do not change the sign of $f(x)f'(x)f''(x)f'''(x)$.) Now $f''(x) > 0$ implies that $f'(x)$ is increasing, and $f'''(x) > 0$ implies that $f'(x)$ is convex, so that $f'(x+a) > f'(x) + af''(x)$ for all x and a. By letting a increase in the latter inequality, we see that $f'(x+a)$ must be positive for sufficiently large a; it follows that $f'(x) > 0$ for all x. Similarly, $f'(x) > 0$ and $f''(x) > 0$ imply that $f(x) > 0$ for all x. Therefore $f(x)f'(x)f''(x)f'''(x) > 0$ for all x, and we are done.