1. Show that 
\[ \sum_{n \in A} \frac{1}{n} < \infty, \]
where \( A \) is the set of positive integers that do not contain a '9' in their decimal expansion.

**Proof.** Let 
\[ S_n = \sum_{m \in A \text{ with } 10^{n-1} \leq m < 10^n} \frac{1}{m}. \]
Since the terms in the sum are positive, the sum is the same as \( \sum_{n \to \infty} S_n \). Now we count the number of terms summed up in \( S_n \). These are the number of natural numbers between \( 10^{n-1} \) and \( 10^n \) with no '9' in their decimal expansion. There are exactly \( n \) digits in any such natural number. Since there are no '9's and since the first digit cannot be 0, there are \( 8 \times 9^{n-1} \) possible choices. For each of these choices, since \( m \geq 10^{n-1} \), we have \( \frac{1}{m} \leq \frac{1}{10^{n-1}} \). Therefore 
\[ S_n \leq \frac{8 \times 9^{n-1}}{10^{n-1}} = 8\left(\frac{9}{10}\right)^{n-1}. \]
Since \( \frac{9}{10} < 1 \), the series \( \sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^{n-1} \) converges as a geometric series. By comparison, the sum \( \sum_{n=1}^{\infty} S_n \) also converges.

2. Evaluate 
\[ \sum_{n=0}^{\infty} \text{Arc cot}(n^2 + n + 1), \]
where \( \text{Arc cot}(t) \) for \( t \geq 0 \) denotes the number \( \theta \) in the interval \( 0 < \theta \leq \pi/2 \) with \( \cot \theta = t \).

A3 1986. There aren’t many series that are easy to evaluate. Typically, to evaluate a series it is geometric, telescoping, or has a nice relation to a known Taylor series. The 1st and 3rd option don’t seem hopeful, but maybe this is a telescoping series in disguise. Recall the difference formulae for \( \sin \) and \( \cos \)
\[ \sin(A - B) = \sin(A) \cos(B) - \cos(A) \sin(B) \]
\[ \cos(A - B) = \cos(A) \cos(B) + \sin(A) \sin(B). \]
Thus
\[ \cot(A - B) = \frac{\cos(A - B)}{\sin(A - B)} = \frac{\cos(A) \cos(B) + \sin(A) \sin(B)}{\sin(A) \cos(B) - \cos(A) \sin(B)} = \frac{\cot(A) \cot(B) + 1}{\cot(B) - \cot(A)}. \]
Setting \( A = \cot^{-1}(a) \) and \( B = \cot^{-1}(b) \) gives us
\[ \cot^{-1}(a) - \cot^{-1}(b) = \cot^{-1}\left(\frac{ab + 1}{b - a}\right). \]
Setting \( b = n + 1 \) and \( a = n \) gives us
\[ \cot^{-1}(n) - \cot^{-1}(n + 1) = \cot^{-1}(n^2 + n + 1). \]
Since \( \cot^{-1}(n) \to 0 \) as \( n \to \infty \), our sum is telescoping and the sum is just \( \cot^{-1}(0) = \frac{\pi}{2} \). \( \square \)
3. A not uncommon calculus mistake is to believe that the product rule for derivatives says that \((fg)' = f'g'\). If \(f(x) = e^{x^2}\), determine, with proof, whether there exists an open interval \((a, b)\) and a non-zero function \(g\) defined on \((a, b)\) such that the wrong product rule is true for \(x\) in \((a, b)\).

**Proof.** Let \(f(x) = e^{x^2}\). The equation \((fg)' = f'g'\) becomes
\[
e^{x^2} 2xg(x) + e^{x^2} g'(x) = e^{x^2} 2xyg'(x).
\]
Dividing by \(e^{x^2}\) gives
\[
2xg(x) = (2x - 1)g'(x).
\]
Or just
\[
1 + \frac{1}{2x - 1} = \frac{g'(x)}{g(x)}.
\]
Integrating both sides with respect to \(x\) gives
\[
x + \frac{\ln|2x - 1|}{2} = \ln|g(x)| + C,
\]
for some constant \(C\). Hence we can take
\[
g(x) = e^{x\sqrt{2x - 1}},
\]
and consider the interval \((\frac{1}{2}, \infty)\), where \(2x - 1 > 0\). Then
\[
g'(x) = e^{x\sqrt{2x - 1}} + e^x \frac{1}{\sqrt{2x - 1}} = e^{x\sqrt{2x - 1}}(1 + \frac{1}{2x - 1}).
\]
Therefore
\[
(fg)' = \frac{d}{dx} \left( e^{x^2 + 1\sqrt{2x - 1}} \right) = e^{x^2 + x} \frac{1}{\sqrt{2x - 1}} + e^{x^2 + x}(2x + 1)\sqrt{2x - 1} = e^{x^2 + x}(2x + 1 + \frac{1}{2x - 1})
\]
and
\[
f'g' = e^{x^2}xe^x\sqrt{2x - 1}(1 + \frac{1}{2x - 1}) = e^{x^2 + x}(2x + 1 + \frac{1}{2x - 1}).
\]

4. Find all real-valued continuously differentiable functions \(f\) on the real line such that for all \(x\),
\[
(f(x))^2 = \int_0^x [(f(t))^2 + (f'(t))^2] dt + 1990.
\]

**Solution [B-1 1990]**
Putting \(y = f(x)\) and differentiating the relationship gives \(2yy' = y^2 + (y')^2\) or \((y - y')^2 = 0\). So \(y = y'\). Integrating, \(y = Ae^x\). But \(y(0) = \pm\sqrt{1990}\), so \(f(x) = \pm\sqrt{1990e^x}\). The function is continuous, so we cannot "mix" the two solutions: either \(f(x) = \sqrt{1990}e^x\) for all \(x\), or \(f(x) = -\sqrt{1990}e^x\) for all \(x\).
5. Let \( f \) be a real function on the real line with continuous third derivative. Prove that there exists a point \( a \) such that

\[
f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.
\]

Solution [A-3 (1998)]

If at least one of \( f(a) \), \( f'(a) \), \( f''(a) \), or \( f'''(a) \) vanishes at some point \( a \), then we are done. Hence we may assume each of \( f(x) \), \( f'(x) \), \( f''(x) \), and \( f'''(x) \) is either strictly positive or strictly negative on the real line. By replacing \( f(x) \) by \( -f(x) \) if necessary, we may assume \( f''(x) > 0 \); by replacing \( f(x) \) by \( f(-x) \) if necessary, we may assume \( f'''(x) > 0 \). (Notice that these substitutions do not change the sign of \( f(x)f'(x)f''(x)f'''(x) \).) Now \( f''(x) > 0 \) implies that \( f'(x) \) is increasing, and \( f'''(x) > 0 \) implies that \( f'(x) \) is convex, so that \( f'(x + a) > f'(x) + af''(x) \) for all \( x \) and \( a \). By letting \( a \) increase in the latter inequality, we see that \( f'(x + a) \) must be positive for sufficiently large \( a \); it follows that \( f'(x) > 0 \) for all \( x \). Similarly, \( f'(x) > 0 \) and \( f''(x) > 0 \) imply that \( f(x) > 0 \) for all \( x \). Therefore \( f(x)f'(x)f''(x)f'''(x) > 0 \) for all \( x \), and we are done.