Putnam Practice Set #2

1. Let n be an even positive integer, and let $p(x)$ be an n-degree polynomial such that $p(-k) = p(k)$ for $k = 1, 2, ..., n$. Prove that there is a polynomial $q(x)$ such that $p(x) = q(x^2)$.

   Proof. Let $p(x) = a_0 + a_1x + \cdots a_nx^n$. The $(n-1)$-degree polynomial $p(x) - p(-x) = 2(a_1x + a_3x^3 + \cdots + a_{n-1}x^{n-1})$ vanishes at n different points, hence it must be identically null; i.e., $a_1 = a_3 = \cdots = a_{n-1} = 0$. Hence $p(x) = a_0 + a_2x^2 + \cdots a_nx^n$, and $q(x) = a_0 + a_2x + a_4x^2 + \cdots + a_nx^{n/2}$. □

2. Let $f(x)$ and $g(x)$ be nonzero polynomials with real coefficients such that $f(x^2 + x + 1) = f(x)g(x)$. Show that $f(x)$ has even degree.

   Proof. First we prove (by contradiction) that $f(x)$ has no real roots. In fact, if x_1 is a real root of $f(x)$, then we have that $x_2 = x_1^2 + x_1 + 1$ is also a real root of $f(x)$ since $f(x_1^2 + x_1 + 1) = f(x_1)g(x_1) = 0$. But $x_1^2 + 1 > 0$, hence $x_2 = x_1^2 + x_1 + 1 > x_1$. Repeating the reasoning we get that $x_3 = x_2^2 + x_2 + 1$ is another root larger than x_2, and so on, so we get an infinite increasing sequence of roots, which is impossible. Consequently $f(x)$ must have even degree, since odd degree polynomials have at least one real root. □

3. The product of two of the four zeros of the quartic equation

$$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$$

   is -32. Find k.

   Proof. Let the zeros be a, b, c, d. The relationship between zeros and coefficients yields

$$a + b + c + d = 18$$

$$ab + ac + ad + bc + bd + cd = k$$

$$abc + abd + acd + bcd = -200$$

$$abcd = -1984.$$

   Assume $ab = -32$ and let $u = a + b$, $v = c + d$, $w = cd$. Then

$$u + v = 18$$

$$-32 + uv + w = k$$

$$-32v + uw = -200$$

$$-32w = -1984.$$

   From the last equation we get that $w = 62$. Replacing in the other equations we get $u = 4$ and $v = 14$. Hence

$$k = -32 + 4 * 14 + 62 = 86.$$

   □

4. A polynomial $f(x) = x^4{+}?x^3{+}?x^2{+}?x + 1$ has three undetermined coefficients denoted by "?". The players A and B move alternately, replacing a question mark by a real number until all question marks are replaced. A wins if all zeros of the polynomial are complex. B wins if at least one zero is real. Show that if B is allowed to pick the coefficient of x^2, then he can win.

Proof. Given thet $f(x) \to +\infty$ as $x \to \pm\infty$, $f(x)$ will have a root if $f(x) \leq 0$ for some x.

Let a, b, c be the coefficients picked, so that the polynomial becomes $x^4 + ax^3 + bx^2 + cx + 1$. Then we have

$$f(x)f(-x) = x^4 + ax^3 + bx^2 + cx + 1 = (x^4 + bx^2 + 1)^2 - (ax^2 + c)^2x^2.$$

Using the quadratic formula,

$$\alpha = \sqrt{\frac{-b + \sqrt{b^2 - 4}}{2}}$$

is a real root of $x^4 + bx^2 + 1$, as long as $b \leq -2$. In that case,

$$f(\alpha)f(-\alpha) = -(a\alpha^2 + c)^2\alpha^2 \leq 0.$$

Therefore $f(\alpha) \leq 0$ or $f(-\alpha) \leq 0$, and player B wins. □

5. Let k be a positive integer. Find all polynomials $p(x)$ with real coefficients such that

$$p(p(x)) = p(x)^k.$$

Proof. A polynomial is either constant or it takes on infinitely many values. If $p(x) = c$, then $c = c^k$, so either $k = 1$ and c is arbitrary or $k > 1$ and $c = 0$ or $k > 1$ and $c = 0$ or 1 or possibly -1 if k is odd. For non-constant $p(x)$, let $z = p(x)$. We then have $p(z) = z^k$ for infinitely many values of z. Hence the polynomial $p(z) - z^k$ has infinitely many roots but a finite degree, so the polynomial must be zero everywhere, or $p(z) - z^k = 0$, so that $p(x) = x^k$ is the only family of non-constant polynomials which solve the given equation. □