ISU Putnam Practice Set 2

Wednesday, February 3, 2021

1. Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

   (i) $A_1 \cup A_2 \cup A_3 = \{1,2,3,4,5,6,7,8,9,10\}$, and

   (ii) $A_1 \cap A_2 \cap A_3 = \emptyset$.

   Solution (Putnam 1985, A1):

   The question asks for the number of ways of placing the integers 1 through 10 in the Venn diagram of three sets A_1, A_2, A_3, such that no number is placed in the region $A_1 \cap A_2 \cap A_3$. Since there are 6 other regions in the diagram, the answer is 6^{10}.

2. An *inversion* in a permutation σ is a pair (i, j) where $i < j$ and $\sigma(i) > \sigma(j)$.

   Consider the permutations

$$\sigma_1 = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & 19 & 20 \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_{19} & a_{20} \end{bmatrix},$$

$$\sigma_2 = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & 19 & 20 \\ a_{19} & a_{20} & a_{17} & a_{18} & \cdots & a_1 & a_2 \end{bmatrix}.$$

   Prove that if σ_1 has 100 inversions, then σ_2 has at most 100 inversions.

   Solution (1979 Romanian Mathematical Olympiad):

   Let see what happens to an inversion (i, j) (where $i < j$ and $a_i > a_j$) of σ_1 in σ_2. If i, j have the same parity then a_i and a_j are switched in σ_2, so (i, j) is not longer an inversion in σ_2. Similarly, if i is even and j is odd then a_i, a_j are also switched in σ_2, so (i, j) is not longer an inversion in σ_2.

   It is left to consider the case i odd and j is even. If $j > i+1$ then again a_i, a_j are switched in σ_2, so (i, j) is not longer an inversion in σ_2. However, if i, j are consecutive then the pair is not permuted in σ_2, so the inversion is preserved. There are at most 10 such pairs $((1,2),(3,4),\ldots,(19,20))$, so at most 10 inversions are "transmitted" from σ_1 to σ_2. Since

σ_1 has 100 inversions, and at most 10 become inversion also in σ_2, we have that at least 90 inversions are "lost": they are no longer inversions in σ_2.

It follows that out of the $\binom{20}{2} = 190$ pairs (i, j), $i < j$ at least 90 are not inversions in σ_2, so at most $190 - 90 = 100$ are inversions in σ_2.

3. Given 2^{n-1} subsets of a set with n elements with the property that any three have nonempty intersection, prove that the intersection of all the sets is nonempty.

   Solution (1971 German Mathematical Olympiad):

   Let $S = \{A_1, \dots, A_{2^{n-1}}\}$ be the family of subsets of a set T with n elements. Note that every two sets in S intersect, so S cannot contain a subset A and its complement A^c. However, since S has 2^{n-1} elements, it must contain either A or A^c for every subset A of T. So for every $A_i, A_j \in S$, either $A_i \cap A_j \in S$ or $(A_i \cap A_j)^c \in S$. But the latter case is not possible since $A_i \cap A_j \cap (A_i \cap A_j)^c = \emptyset$, in contradiction to the condition on S. So we have $A_i \cap A_j \in S$ whenever $A_i, A_j \in S$. In particular, $A_i \cap A_j \neq \emptyset$.

   Now we can show by induction on k that every k sets in S have non-empty intersection. The argument above proves this for $k = 2$. The step is similar (replace A_j by the non-empty intersection of $k-1$ subsets, that follows by the induction hypothesis).

4. The sequence of digits

   $$1234567891011121314151617181920\dots$$

   is obtained by writing the positive integers in order. If the 10^nth digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example, $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(2021)$.

   Solution (Putnam 1987, A2):

   Let $g(m)$ denote the total number of digits in the integers with m or fewer digits. Then $f(n)$ equals the integer m such that $g(m-1) < 10^n \leq g(m)$. There are $10^r - 10^{r-1}$ numbers with exactly r digits, so $g(m) = \sum_{r=1}^{m} r(10^r - 10^{r-1})$.

We have

$$g(2017) = \sum_{r=1}^{2017} r(10^r - 10^{r-1}) < \sum_{r=1}^{2017} 2017(10^r - 10^{r-1}) < 2017 \cdot 10^{2017} < 10^{2021},$$

and

$$g(2018) = \sum_{r=1}^{2018} r(10^r - 10^{r-1}) > 2018(10^{2018} - 10^{2017}) = 2018 \cdot 9 \cdot 10^{2017} > 10^{2021}.$$

So $f(2021) = 2018$.

5. Let F be a field in which $1+1 \neq 0$. Show that the solutions to the equation $x^2+y^2=1$ with x,y in F is given by $(x,y)=(1,0)$ and

$$(x,y) = \left(\frac{r^2-1}{r^2+1}, \frac{2r}{r^2+1}\right)$$

where r runs through the elements of F such that $r^2 \neq -1$.

Solution (Putnam 1987, B3):

It is easy to check that $(x,y)=(1,0)$ and $(x_r,y_r) = \left(\frac{r^2-1}{r^2+1}, \frac{2r}{r^2+1}\right)$ when $r^2 \neq -1$ are solutions. Conversely, suppose that $x^2+y^2=1$. If $x=1$ then $y=0$ and we get the solution $(x,y)=(1,0)$. Otherwise, $x \neq 1$. Define $r = \frac{y}{1-x}$ (the motivation for this definition is that $r = \frac{y_r}{1-x_r}$). Then

$$1-x^2 = y^2 = r^2(1-x)^2,$$

and since $x \neq 1$ we may divide by $(1-x)$ to get

$$1+x = y^2 = r^2(1-x),$$

implying

$$(r^2+1)x = r^2-1.$$

If $r^2=-1$ then this equation is $-2=0$, contradicting $1+1 \neq 0$. So $r^2 \neq -1$, and thus $x = \frac{r^2-1}{r^2+1} = x_r$ and $y = r(1-x) = \frac{2r}{r^2+1} = y_r$. Hence every solution to the equation $x^2+y^2=1$ not equal to $(1,0)$ is of the form (x_r,y_r) for some $r \in F$ such that $r^2 \neq -1$.