ISU Putnam Practice Set 2

Wednesday, February 3, 2021

1. Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

   (i) $A_1 \cup A_2 \cup A_3 = \{1,2,3,4,5,6,7,8,9,10\}$, and

   (ii) $A_1 \cap A_2 \cap A_3 = \emptyset$.

2. An *inversion* in a permutation σ is a pair (i, j) where $i < j$ and $\sigma(i) > \sigma(j)$.

   Consider the permutations

$$\sigma_1 = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & 19 & 20 \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_{19} & a_{20} \end{bmatrix},$$

$$\sigma_2 = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & 19 & 20 \\ a_{19} & a_{20} & a_{17} & a_{18} & \cdots & a_1 & a_2 \end{bmatrix}.$$

   Prove that if σ_1 has 100 inversions, then σ_2 has at most 100 inversions.

3. Given 2^{n-1} subsets of a set with n elements with the property that any three have nonempty intersection, prove that the intersection of all the sets is nonempty.

4. The sequence of digits

$$1234567891011121314151617181920\ldots$$

   is obtained by writing the positive integers in order. If the 10^nth digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example, $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(2021)$.

5. Let F be a field in which $1 + 1 \neq 0$. Show that the solutions to the equation $x^2 + y^2 = 1$ with x, y in F is given by $(x, y) = (1, 0)$ and

$$(x, y) = \left(\frac{r^2 - 1}{r^2 + 1}, \frac{2r}{r^2 + 1}\right)$$

   where r runs through the elements of F such that $r^2 \neq -1$.