1. Determine, with proof, the number of ordered triples \((A_1, A_2, A_3)\) of sets which have the property that

(i) \(A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\), and

(ii) \(A_1 \cap A_2 \cap A_3 = \emptyset\).

2. An inversion in a permutation \(\sigma\) is a pair \((i, j)\) where \(i < j\) and \(\sigma(i) > \sigma(j)\).

Consider the permutations

\[
\sigma_1 = \begin{bmatrix}
1 & 2 & 3 & 4 & \cdots & 19 & 20 \\
a_1 & a_2 & a_3 & a_4 & \cdots & a_{19} & a_{20}
\end{bmatrix},
\]

\[
\sigma_2 = \begin{bmatrix}
1 & 2 & 3 & 4 & \cdots & 19 & 20 \\
a_{19} & a_{20} & a_{17} & a_{18} & \cdots & a_1 & a_2
\end{bmatrix}.
\]

Prove that if \(\sigma_1\) has 100 inversions, then \(\sigma_2\) has at most 100 inversions.

3. Given \(2^{n-1}\) subsets of a set with \(n\) elements with the property that any three have nonempty intersection, prove that the intersection of all the sets is nonempty.

4. The sequence of digits

\[
1234567891011121314151617181920\ldots
\]

is obtained by writing the positive integers in order. If the 10\(^{th}\) digit in this sequence occurs in the part of the sequence in which the \(m\)-digit numbers are placed, define \(f(n)\) to be \(m\).

For example, \(f(2) = 2\) because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, \(f(2021)\).

5. Let \(F\) be a field in which \(1 + 1 \neq 0\). Show that the solutions to the equation \(x^2 + y^2 = 1\) with \(x, y\) in \(F\) is given by \((x, y) = (1, 0)\) and

\[
(x, y) = \left(\frac{r^2 - 1}{r^2 + 1}, \frac{2r}{r^2 + 1}\right)
\]

where \(r\) runs through the elements of \(F\) such that \(r^2 \neq -1\).