1. Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$

Proof. We prove this for any fraction $\frac{a}{b}$ by induction the the largest prime divisor of a or b. In the base case, $a = b = 1$. We may write $\frac{a}{b} = \frac{2!}{2!}$.

Now suppose a, b have no common prime divisors and let p be the largest prime divisor of a or b. WLOG, we assume $p|a$, for otherwise we can take the reciprocal, apply the following argument and go back. Since $p|a$, we can write $\frac{a}{b} = \frac{(p!)^r a'}{((p-1)!)^r b}$ for some integer r such that a' and $((p-1)!)^r b$ have prime divisors strictly smaller than p. By induction, $\frac{a'}{((p-1)!)^r b}$ can be written as a quotient of factorials of primes. Multiplying this representation by $(p!)^r$ gives the required representation of $\frac{a}{b}$ as a quotient of prime factorials.

□

2. Prove the following equality for all natural numbers n:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$

Proof. We prove this by induction on n. For $n = 1$, we have the statement

$$1 - \frac{1}{2} = \frac{1}{2},$$

which is clearly true. Now assume the above equation is true for all $n <= N$. Then

$$\begin{aligned} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2(N+1)+1} - \frac{1}{2(N+1)} &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots + \tfrac{1}{2N-1} - \tfrac{1}{2N} + \tfrac{1}{2N+1} - \tfrac{1}{2N+2} \\ &= \tfrac{1}{N+1} + \tfrac{1}{N+2} + \cdots + \tfrac{1}{2N} + \tfrac{1}{2N+1} - \tfrac{1}{2N+2} \\ &= \tfrac{1}{N+2} + \cdots + \tfrac{1}{2N} + \tfrac{1}{2N+1} + \tfrac{2}{2N+2} - \tfrac{1}{2N+2} \\ &= \tfrac{1}{N+2} + \cdots + \tfrac{1}{2N} + \tfrac{1}{2N+1} + \tfrac{1}{2N+2}, \end{aligned}$$

where the second equality uses the inductive hypothesis. Therefore, the equation is true for all n by induction. □

3. In a room there are 10 people, each of which has age between 1 and 60 (ages are only integers). Prove that among them there are 2 groups of people, with no common person, the sum of whose ages is the same.

Proof. Since there the ages are at most 60, the sum of any subgroup of the 10 people can be at most 600. There are $2^{10} = 1024$ distinct subsets of the 10 people, so at least 2 of them must have a common age-sum - call these two sets A and B. If $A \cap B =$, we are done. Otherwise, we take $A - B$ and $B - A$ as our sets. Since we have taken away the members of $A \cap B$, the age sum is the same and nonzero for both.

□

4. Define the Fibonacci sequence as $F_1 = 1, F_2 = 1$, and in general, $F_{n+2} = F_{n+1} + F_n$ for $n1$. (So, e.g., $F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21$, and so on.) (a) Prove that every third Fibonacci number is even, and the rest are odd. (b) More generally, prove that F_k divides F_{nk} for any n and k positive integers.

   Proof. (a) $F_1 = 1$ and $F_2 = 1$ are odd, $F_3 = 2$ is even. We claim that F_{3n+1} and F_{3n+2} are odd and F_{3n+3} is even for all natural numbers n. Note that

$$F_{3n+1} = F_{3n} + F_{3n-1}$$

$$F_{3n+2} = F_{3n+1} + F_{3n}$$

$$F_{3n+3} = F_{3n+1} + F_{3n+2}.$$

   By induction, we have that F_{3n} is even and F_{3n-1} is odd. Therefore F_{3n+1} is odd. Since F_{3n} is even and F_{3n+1} is odd, so is F_{3n+2}. It follows that F_{3n+3} is even. By induction, we are done.

   (b) Claim: For all natural numbers $n \geq 1$ and $a \geq 2$, $F_a F_n + F_{a-1} F_{n+1} = F_{a+n+1}$.

   Proof of claim: We use induction on a. When $a = 2$, this reduces to the statement that $1F_n + 1F_{n+1} = F_{n+2}$ for all $n \geq 1$, which is true by the definition of the Fibonacci sequence. Suppose the claim is true for $a < A$. Then

$$\begin{aligned} F_A F_n + F_{A-1} F_{n+1} &= (F_{A-1} + F_{A-2}) F_n + (F_{A-3} + F_{A-2}) F_{n+1} \\ &= F_{A-2} F_n + F_{A-3} F_{n+1} + F_{A-1} F_{n+1} + F_{A-2} F_{n+1} \\ &= F_{A+n-1} + F_{A+n} \\ &= F_{A+n+1}. \end{aligned}$$

   Hence we are done by induction on A.

   Now, using the claim, fix natural numbers k, n. Then if $k = 1$, then $F_k = F_1 = 1$ divides F_n for all n. If $k \geq 2$, then by the claim $F_{nk} = F_k F_{nk-k-1} + F_{k-1} F_{nk-k}$. By induction on n, we can assume that F_k divides F_{nk-k}. Clearly F_k divides F_k. Hence F_k divides F_{nk}. We are done by induction on n.

   □

5. A battle ship is travelling on the number line. It starts at an unknown integer and moves at an unknown constant integer speed (integers per second.) You can fire a cannon once every second at an integer, destroying the ship if it is there. Come up with an algorithm for firing that is guaranteed to destroy the ship in a finite amount of time.

Proof. If v is the velocity of the battleship and s is the starting position, then at time t, the battleship is at position $s + tv$. So if we correctly guess both the velocity and starting position, we can correctly guess the battleship's position. Note that the set $\mathbb{Z} \times \mathbb{Z}$ is countable. Hence there is a function $(V, S) : \mathbb{N} \to \mathbb{Z} \times \mathbb{Z}$ which is bijective. So at time t, if we fire at position $S(t) + tV(t)$, we are guaranteed to eventually hit the battleship.

□

6. Take five arbitrary points on the surface of a sphere. Prove that there is a closed hemisphere (including the boundary) which contains at least four points.

   Proof. Note that any two points on the sphere determine a great circle dividing the sphere into 2 hemispheres. Choose any two of the five points and consider the great circle they determine. Of the remaining 3 points, at least 2 must lie in one of the hemispheres. Hence one closed hemisphere contains at least 4 points.

   □