

2023 - ISU Putnam Practice Set 11

Thursday, November 16, 2023

B-65

1. Evaluate

$$\lim_{n \rightarrow \infty} \int_0^1 \int_0^1 \cdots \int_0^1 \cos^2 \left(\frac{\pi}{2n} (x_1 + x_2 + \cdots + x_n) \right) dx_1 dx_2 \cdots dx_n.$$

2. In a round-robin tournament with n players P_1, P_2, \dots, P_n (where $n > 1$), each player plays one game with each of the other players and the rules are such that no ties can occur. Let w_i and ℓ_i be the number of games won and lost, respectively, by P_i . Show

$$\sum_{i=1}^n w_i^2 = \sum_{i=1}^n \ell_i^2.$$

3. Prove that there are exactly three right-angled triangles whose sides are integers while the area is numerically equal to twice the perimeter.

4. Consider collections of unordered pairs of V different objects a, b, c, \dots . Three pairs such as bc, ca, ab are said to form a triangle. Prove that, if $4E \leq V^2$, it is possible to choose E pairs so that no triangle is formed.

5. If A, B, C, D are four distinct points such that every circle through A and B intersects (or coincides with) every circle through C and D , prove that the four points are either collinear (all of one line) or concyclic (all on one circle).