2023 - ISU Putnam Practice Set 10

Thursday, November 9, 2023

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1. The triangle ABC has an obtuse angle at B, and angle A is less than angle C. The external angle bisector at A meets the line BC at D, and the external angle bisector at B meets the line AC at E. Also, BA = AD = BE. Find angle A.

2. Let α and β be positive real numbers such that $1/\alpha + 1/\beta = 1$. Prove that the line $mx + ny = 1$ with m, n positive reals is tangent to the curve $x^\alpha + y^\alpha = 1$ in the first quadrant $(x, y \geq 0)$ iff $m^\beta + n^\beta = 1$.

3. Show that, for any positive integer n,

$$\sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \left(\frac{n-2r}{n} \binom{n}{r} \right)^2 = \frac{1}{n} \binom{2n-2}{n-1},$$

   where $\lfloor x \rfloor$ means the greatest integer not exceeding x, and $\binom{n}{r}$ is the binomial coefficient "n choose r," with the convention $\binom{n}{0} = 1$.

4. S and T and finite sets. U is a collection of ordered pairs (s,t) with $s \in S$ and $t \in T$. There is no element $s \in S$ such that all possible pairs $(s,t) \in U$. Every element $t \in T$ appears in at least one element of U. Prove that we can find distinct $s_1, s_2 \in S$ and distinct $t_1, t_2 \in T$ such that $(s_1,t_1), (s_2,t_2) \in U$, but $(s_1,t_2), (s_2,t_1) \notin U$.

5. How many possible bijections f on $\{1, 2, \ldots, n\}$ are there such that for each $i = 2, 3, \ldots, n$ we can find $j < n$ with $f(i) - f(j) = \pm 1$?