

2021 - ISU Putnam Practice Set 10

Wednesday, December 2, 2022

B1 Bingo!

1. For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x . Prove that for any two partitions π and π' , there are two distinct numbers x and y in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi'(y)$ and $\pi'(x) = \pi(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S .]

2. Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.

3. Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F . Evaluate $\lim_{\theta \rightarrow 0} |EF|$.
4. Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

5. What is the maximum number of rational points that can lie on a circle in \mathbb{R}^2 whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)