2021 - ISU Putnam Practice Set 10

Wednesday, December 1, 2021

Graphs

1. Three conflicting neighbors have three common wells. Can one draw nine paths con- necting each of the neighbors to each of the wells such that no two paths intersect?

2. In a tournament 2n teams took part. On the first day, n pairs of teams competed. On the second day, other n pairs of teams competed. Show that at the end of the second day one can find n teams such that no two have competed with each other yet.

3. Let G be a connected graph with k edges. Show that it is possible to label the edges of this graph with the numbers 1, 2, . . . , k, so that for every vertex that belongs to at least two edges, the greatest common divisor of the integers that label the edges containing this vertex is equal to 1.

4. Consider a convex polyhedron whose faces are triangles and whose edges are oriented. A singularity is a face whose edges form a cycle, a vertex that belongs only to incoming edges, or a vertex that belongs only to outgoing edges. Show that the polyhedron has at least two singularities.

5. Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game:

   > Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

   Show that the player who signs first will always win by playing as well as possible.