ISU Putnam Practice Set 1 - Solutions

Wednesday, January 27, 2021

1. Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

   Solution (1986 A1): The inequality $x^4 + 36 \leq 13x^2$ is equivalent to $(x-3)(x-2)(x+2)(x+3) \leq 0$ which is satisfied if and only if $x \in [-3,-2] \cup [2,3]$. $f'(x) = 3x^2 - 3 = 0$ if and only if $x = \pm 1$, so the maximum for f must occur at one of the four endpoints. Plugging in we see that the maximum value is $18 = f(3) = f(-2)$.

2. How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

   Solution (1989 A1): There is only one such prime: 101. To see why, suppose $N = 101\cdots0101$ with k ones for some $k \geq 2$. Then

$$99N = 999\cdots9999 = 10^{2k} - 1 = (10^k + 1)(10^k - 1).$$

   If moreover N is prime, then N divides either $10^k + 1$ or $10^k - 1$, and hence one of $\frac{99}{10^k-1} = \frac{10^k+1}{N}$ and $\frac{99}{10^k+1} = \frac{10^k-1}{N}$ is an integer. For $k > 2$, $10^k - 1$ and $10^k + 1$ are both greater than 99, so we get a contradiction. Therefore $k = 2$ and $N = 101$ (which is prime).

3. Let D_n denote the value of the $(n-1) \times (n-1)$ determinant

$$\begin{bmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{bmatrix}.$$

   Is the set $\left\{\frac{D_n}{n!}\right\}_{n \geq 2}$ bounded?

   Solution (1992 B5): The determinant is unchanged when we add one row/column to

another. Subtracting the top row from all others leaves the matrix:

$$\begin{bmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ -2 & 3 & 0 & 0 & \cdots & 0 \\ -2 & 0 & 4 & 0 & \cdots & 0 \\ -2 & 0 & 0 & 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & 0 & 0 & 0 & \cdots & n \end{bmatrix}.$$

For $2 \le i \le n-1$**, add** $2/(i+1)$ **times the** i**th column to the first column to obtain**

$$\begin{bmatrix} 3+\frac{2}{3}+\frac{2}{4}+\frac{2}{5}+\cdots+\frac{2}{n} & 1 & 1 & 1 & \cdots & 1 \\ 0 & 3 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 4 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \end{bmatrix}.$$

This is an upper triangular matrix so its determinant is the product of the diagonal entries, which is

$$D_n = n!(1+\frac{1}{2}+\cdots+\frac{1}{n}).$$

Thus $\frac{D_n}{n!}$ **is the** n**th partial sum of the harmonic series, which is unbounded.**

4. Find the smallest positive integer n such that for every integer m with $0 < m < 1993$, there exists an integer k for which

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}.$$

Solution (1993 B1): It is easy to check that for any positive integers a,b,c,d **with** $\frac{a}{b} < \frac{c}{d}$**, we have**

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

Therefore

$$\frac{m}{1993} < \frac{2m+1}{3987} < \frac{m+1}{1994}$$

for any integer m **with** $0 < m < 1993$**. We show that** 3987 **is the best possible. If**

$$\frac{1992}{1993} < \frac{k}{n} < \frac{1993}{1994},$$

then

$$\frac{1}{1993} > \frac{n-k}{n} > \frac{1}{1994},$$

and so

$$1993 < \frac{n}{n-k} < 1994.$$

Since $k < n$**,** $n-k \neq 1$**. Thus** $n-k \geq 2$**. Thus** $n > 1993(n-k) \geq 3986$**; i.e.** $n \geq 3987$**.**

5. Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

   Solution (1994 A1): Suppose the sum converges to a finite number L**. For** $m \geq 1$**, write**

$$b_m = \sum_{i=2^{m-1}}^{2^m-1} a_i,$$

   so that $L = \sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} b_i$**. Summing** $a_n \leq a_{2n} + a_{2n+1}$ **from** $n = 2^{m-1}$ **to** $n = 2^m - 1$ **yields** $b_m \leq b_{m+1}$**. Therefore** $\lim_{m\to\infty} b_m \neq 0$ **and hence** $\sum b_m$ **diverges - a contradiction.**