Generating Functions

1. Find the general-term formula for the sequence \((y_n)_{n \geq 0}\) with \(y_0 = 1\) and \(y_n = ay_{n-1} + b^n\) for \(n \geq 1\), where \(a, b\) are fixed distinct real numbers.

2. Compute the sums
   \[
   \sum_{k=0}^{n} k \binom{n}{k} \quad \text{and} \quad \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k}.
   \]

3. Denote by \(P(n)\) the number of partitions of the positive integer \(n\); i.e. the number of ways of writing \(n\) as a sum of positive integers. Prove that the generating function of \(P(n), n \geq 1\) is
   \[
   \sum_{n=0}^{\infty} P(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots}.
   \]

4. Prove that the number of ways of writing \(n\) has a sum of distinct positive integers is equal to the number of ways of writing \(n\) as a sum of odd positive numbers.

5. Let \(p\) be an odd prime number. Find the number of subsets of \(\{1, 2, \ldots, p\}\) with the sum of the elements divisible by \(p\).