2021 - ISU Putnam Practice Set 9

Wednesday, November 24, 2021

Generating Functions

1. Find the general-term formula for the sequence $(y_n)_{n\geq 0}$ with $y_0 = 1$ and $y_n = ay_{n-1} + b^n$ for $n \geq 1$, where a, b are fixed distinct real numbers.

2. Compute the sums
$$\sum_{k=0}^{n} k\binom{n}{k} \text{ and } \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}.$$

3. Denote by $P(n)$ the number of partitions of the positive integer n; i.e. the number of ways of writing n as a sum of positive integers. Prove that the generating function of $P(n)$, $n \geq 1$ is
$$\sum_{n=0}^{\infty} P(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}.$$

4. Prove that the number of ways of writing n has a sum of distinct positive integers is equal to the number of ways of writing n as a sum of odd positive numbers.

5. Let p be an odd prime number. Find the number of subsets of $\{1, 2, \ldots, p\}$ with the sum of the elements divisible by p