2021 - ISU Putnam Practice Set 9 - Solutions

Monday, October 26, 2026

Probability

1. What is the probability that 3 points selected at random on a circle lie on some semicircle?

   Solution: We assume that the circle of the problem is the unit circle centered at the origin O. The space of all possible choices of three points P_1, P_2, P_3 is the product of three circles; the volume of this space is $8\pi^3$. Let us first measure the volume of the configurations (P_1, P_2, P_3) such that the arc $P_1P_2P_3$ is included in a semicircle and is oriented counterclockwise from P_1 to P_3. The condition that the arc is contained in a semicircle translates to $0 \leq angleP_1OP_2 \leq \pi$ and $0 \leq angleP_2OP_3 \leq \pi - angleP_1OP_2$. The point P_1 is chosen randomly on the circle, and for each P_1 the region of the angles θ_1 and θ_2 such that $0 \leq \theta_1\pi$ and $0 \leq \theta_2 \leq \pi - \theta_1$ is an isosceles right triangle with leg equal to π. Hence the region of points (P_1, P_2, P_3) subject to the above constraints has volume $2\pi \cdot \frac{1}{2}\pi^2 = \pi^3$. There are $3! = 6$ such regions and they are disjoint. Therefore, the volume of the favorable region is $6\pi^3$. The desired probability is therefore $\frac{6\pi^3}{8\pi^3} = \frac{3}{4}$.

2. Let $n \geq 4$ be given, and suppose that the points $P_1, P_2, \ldots, P_n$ are randomly chosen on a circle. Consider the convex n-gon whose vertices are these points. What is the probability that at least one of the vertex angles of this polygon is acute?

   Solution: The angle at the vertex P_i is acute if and only if all other points lie on an open semicircle facing P_i. We first deduce from this that if there are any two acute angles at all, they must occur consecutively. Otherwise, the two arcs that these angles subtend would overlap and cover the whole circle, and the sum of the measures of the two angles would exceed 180°. So the polygon has either just one acute angle or two consecutive acute angles. In particular, taken in counterclockwise order, there exists exactly one pair of consecutive angles the second of which is acute and the first of which is not. We are left with the computation of the probability that for one of the points P_j, the angle at P_j is not acute, but the following angle is. This can be done using integrals. But there is a clever argument that reduces the geometric probability to a probability with a finite number of outcomes. The idea is to choose randomly $n-1$ pairs of antipodal points, and then among these to

choose the vertices of the polygon. A polygon with one vertex at P_j and the other among these points has the desired property exactly when $n-2$ vertices lie on the semicircle to the clockwise side of P_j and one vertex on the opposite semicircle. Moreover, the points on the semicircle should include the counterclockwise-most to guarantee that the angle at P_j is not acute. Hence there are $n-2$ favorable choices of the total 2^{n-1} choices of points from the antipodal pairs. The probability for obtaining a polygon with the desired property is therefore $\frac{(n-2)}{2^{n-1}}$. Integrating over all choices of pairs of antipodal points preserves the ratio. The events $j = 1,2,\ldots,n$ are independent, so the probability has to be multiplied by n. The answer to the problem is therefore $\frac{n(n-2)}{2^{n-1}}$.

3. The temperatures in Chicago and Detroit are x° and y°, respectively. These temperatures are not assumed to be independent; namely, we are given the following:

   (a) $P(x^\circ = 70^\circ) = a$

   (b) $P(y^\circ = 70^\circ) = b$

   (c) $P(\max\{x^\circ, y^\circ\} = 70^\circ) = c$.

   Determine $P(\min\{x^\circ, y^\circ\} = 70^\circ)$ in terms of a, b, c.

   Solution Putnam 1968 B1: Let A, B, C, D denote the events $x = 70^\circ$, $y = 70^\circ$, $\max(x^\circ, y^\circ) = 70^\circ$ and $\min(x^\circ, y^\circ) = 70^\circ$. Then $A \cup B = C \cup D$ and $A \cap B = C \cap D$. Hence

$$P(A) + P(B) = P(A \cup B) + P(A \cap B) = P(C \cup D) + P(C \cap D) = P(C) + P(D).$$

   Therefore $P(D) = P(A) + P(B) - P(C) = a + b - c$.

4. What is the probability that a permutation of the first n positive integers has the numbers 1 and 2 within the same cycle?

   Solution: The total number of permutations is of course $n!$. We will count instead the number of permutations for which 1 and 2 lie in different cycles. If the cycle that contains 1 has length k, we can choose the other $k-1$ elements in $\binom{n-2}{k-1}$ ways from the set $\{3,4,...,n\}$. There exist $(k-1)!$ circular permutations of these elements, and $(n-k)!$ permutations of the remaining $n-k$ elements. Hence the total number of permutations for which 1 and 2 belong to different cycles is equal to

$$\sum_{k=1}^{n-1} \binom{n-2}{k-1} (k-1)!(n-k)! = (n-2)! \sum_{k=1}^{n-1} (n-k) = (n-2)! \frac{n(n-1)}{2} = \frac{n!}{2}.$$

It follows that exactly half of all permutations contain 1 and 2 in different cycles, and so half contain 1 and 2 in the same cycle. The probability is $\frac{1}{2}$.

BETTER SOLUTION DUE TO HENRY SIMMONS 11/18/2022:

Let $\sigma \in S_n$ be a permutation. Observe that the function $\sigma \mapsto \sigma \circ (12)$ maps any permutation in which 1 and 2 appear in the same cycle to one that does not, and vice versa. Thus the total number of permutations with 1 and 2 in the same cycle is the same as those without. The probability is $\frac{1}{2}$.

5. An unbiased coin is tossed n times. Find a formula, in closed form, for the expected value of $|H-T|$, where H is the number of heads, and T is the number of tails.

Solution Putnam 1974 A4: There are $\binom{n}{k}$ ways in which exactly k tails appear, and in this case the difference is $n-2k$ Hence the expected value of $|H-T|$ is

$$\frac{1}{2^n}\sum_{k=0}^{n}|n-2k|\binom{n}{k}.$$

Evaluate the sum as follows:

$$\begin{aligned}
\frac{1}{2^n}\sum_{k=0}^{n}|n-2k|\binom{n}{k} &= \frac{1}{2^{n-1}}\sum_{m=0}^{\lfloor n/2\rfloor}(n-2m)\binom{n}{m} \\
&= \frac{1}{2^{n-1}}\left(\sum_{m=0}^{\lfloor n/2\rfloor}(n-m)\binom{n}{m}-\sum_{m=0}^{\lfloor n/2\rfloor}m\binom{n}{m}\right) \\
&= \frac{1}{2^{n-1}}\left(\sum_{m=0}^{\lfloor n/2\rfloor}n\binom{n-1}{m}-\sum_{m=0}^{\lfloor n/2\rfloor}n\binom{n-1}{m-1}\right) \\
&= \frac{n}{2^{n-1}}\binom{n-1}{\lfloor \frac{n}{2}\rfloor}.
\end{aligned}$$