2023 - ISU Putnam Practice Set 7

Thursday, October 19, 2023

Linear Algebra

1. Let A and B be 2×2 matrices with real entries satisfying $(AB-BA)^n = I_2$ for some positive integer n. Prove that n is even and $(AB-BA)^4 = I_2$.

2. Let A and B be 3×3 matrices with real elements such that

$$\det A = \det B = \det(A+B) = \det(A-B) = 0.$$

   Prove that $\det(xA+yB) = 0$ for any real numbers x and y.

3. Let a,b,c,d be positive numbers different from 1, and x,y,z,t real numbers satisfying $a^x = bcd$, $b^y = cda$, $c^z = dab$, $d^t = abc$. Prove that

$$\det\begin{pmatrix} -x & 1 & 1 & 1 \\ 1 & -y & 1 & 1 \\ 1 & 1 & -z & 1 \\ 1 & 1 & 1 & -t \end{pmatrix} = 0.$$

4. M and N are real unequal $n\times n$ matrices satisfying $M^3 = N^3$ and $M^2N = N^2M$. Can we choose M and N so that M^2+N^2 is invertible?

5. Let A and B be 2×2 matrices with integer entries such that A, $A+B$, $A+2B$, $A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.