2021 - ISU Putnam Practice Set 7

Wednesday, October 20, 2021

Polynomials

1. Let $P(x)$ be a polynomial with complex coefficients. Prove that $P(x)$ is an even func- tion if and only if there exists a polynomial $Q(x)$ with complex coefficients satisfying $P(x) = Q(x)Q(-x)$. (Sorry for the repeated problem.)

2. Suppose $p(x) = x^4 + ax^3 + bx^2 + cx + d$ be a polynomial with rational coefficients. Suppose $p(x)$ has exactly one real root r. Prove that r is rational.

3. Let $a \in \mathbb{C}$ and $n \geq 2$. Prove that the polynomial equation $ax^n + x + 1 = 0$ has a root of absolute value less than or equal to 2.

4. Suppose u, v, w, z are complex numbers for which $u + v + w + z = u^2 + v^2 + w^2 + z^2 = 0$. Prove that
$$(u^4 + v^4 + w^4 + z^4)^2 = 4(u^8 + v^8 + w^8 + z^8).$$

5. Let k be the smallest positive integer for which there exist distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial
$$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$
has exactly k nonzero coefficients. Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.