2021 - ISU Putnam Practice Set 6

Wednesday, October 28, 2022

Algebra

1. Let F be a finite field having an odd number m of elements. Let $p(x)$ be an irreducible (i.e. nonfactorable) polynomial over F of the form x^2+bx+c with $b,c \in F$. For how many elements $k \in F$ is $p(x)+k$ irreducible over F?

2. Let b and c be fixed real numbers and let the ten points (j, y_j) $j = 1,2,\ldots,10$ lie on the parabola $y = x^2+bx+c$. For $j = 1,2,\ldots,9$, let I_j be the point of intersection of the tangents to the given parabola at (j, y_j) and $(j+1, y_{j+1})$. Determine the polynomial function $y = g(x)$ of least degree who graph passes through all nine points I_j.

3. Find all polynomials of two variables satisfying

$$P(a,b)P(c,d) = P(ac+bd, ad+bc)$$

   for all real numbers a,b,c,d.

4. Let n be a positive integer, and define

$$f(n) = 1! + 2! + \cdots + n!.$$

   Find polynoials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n)$$

   for all $n \geq 1$.

5. Prove or disprove the following statement: If F is a finite set with two or more elements, then there exists a binary operation $*$ on F such that for all $x,y,z \in F$:

   (a) $x*z = y*z$ implies $x = y$ (right cancelation holds), and

   (b) $x*(y*z) \neq (x*y)*z$ (no case of associativity holds).