2021 - ISU Putnam Practice Set 6

Wednesday, October 13, 2021

Polynomials

1. Find all solutions to the equation

$$(x+1)(x+2)(x+3)^2(x+4)(x+5) = 360.$$

2. Find all polynomials $P(x)$ satisfying the equation

$$(x+1)P(x) = (x-10)P(x+1).$$

3. Let $P(x)$ be a polynomial with complex coefficients. Prove that $P(x)$ is an even function if and only if there exists a polynomial $Q(x)$ with complex coefficients satisfying

$$P(x) = Q(x)Q(-x).$$

4. Let $P(x)$ be a polynomial of degree n. Given that $P(k) = \frac{k}{k+1}$ for $k = 0, 1, \ldots, n$, find $P(m)$ for $m > n$.

5. Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.