2023 - ISU Putnam Practice Set 5

Thursday, October 5, 2023

Recurrence relations

1. Consider the sequence (u_n) defined by $u_0 = u_1 = u_2 = 1$, and

$$\det \begin{pmatrix} u_{n+3} & u_{n+2} \\ u_{n+1} & u_n \end{pmatrix} = n! \qquad \text{for } n \geq 0.$$

   Prove that u_n is an integer for all n.

2. Find the general term of the sequence given by $x_0 = 3$, $x_1 = 4$, and

$$(n+1)(n+2)x_n = 4(n+1)(n+3)x_{n-1} - 4(n+2)(n+3)x_{n-2}, \qquad n \geq 2.$$

3. Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

   (as usual, the function ln is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \cdots$$

   converges and find its sum.

4. The sequence (x_n) is defined by $x_1 = 4$, $x_2 = 19$, and for $n \geq 2$,

$$x_{n+1} = \left\lceil \frac{x_n^2}{x_{n-1}} \right\rceil.$$

   the smallest integer greater than or equal $\frac{x_n^2}{x_{n-1}}$. Prove that $x_n - 1$ is always a multiple of 3.

5. Prove that every nonzero coefficient of the Taylor series of

$$(1 - x + x^2)e^x$$

   about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.