Calculus 1

1. Let \( f(x) = a_1 \sin(x) + a_2 \sin(2x) + \cdots + a_n \sin(nx) \), where \( a_1, a_2, \ldots, a_n \) are real numbers and \( n \) is a positive integer. Given that \( |f(x)| \leq |\sin(x)| \) for all real \( x \), prove that

\[ |a_1 + 2a_2 + \cdots + na_n| \leq 1. \]

2. Prove that not all zeros of the polynomial \( P(x) = x^4 - \sqrt{7}x^3 + 4x^2 - \sqrt{22}x + 15 \) are real.

3. For any real number \( \lambda \geq 1 \), denote by \( f(\lambda) \) the real solution to the equation

\[ x(1 + \ln x) = \lambda. \]

Prove that

\[ \lim_{\lambda \to \infty} \frac{f(\lambda)}{\lambda / \ln(\lambda)} = 1. \]

4. Find all differentiable functions \( f : \mathbb{R} \to \mathbb{R} \) such that

\[ f'(x) = \frac{f(x+n) - f(x)}{n} \]

for all real numbers \( x \) and all positive integers \( n \).

5. For each continuous function \( f : [0, 1] \to \mathbb{R} \), let \( I(f) = \int_0^1 x^2 f(x) \, dx \) and \( J(x) = \int_0^1 x (f(x))^2 \, dx \).

Find the maximum value of \( I(f) - J(f) \) over all such functions \( f \).