2023 - ISU Putnam Practice Set 4

Thursday, September 28, 2023

Calculus 1

1. Let $f(x) = a_1 \sin(x) + a_2 \sin(2x) + \cdots + a_n \sin(nx)$, where $a_1, a_2, \ldots, a_n$ are real numbers and n is a positive integer. Given that $|f(x)| \le |\sin(x)|$ for all real x, prove that

$$|a_1 + 2a_2 + \cdots na_n| \le 1.$$

2. Prove that not all zeros of the polynomial $P(x) = x^4 - \sqrt{7}x^3 + 4x^2 - \sqrt{22}x + 15$ are real.

3. For any real number $\lambda \ge 1$, denote by $f(\lambda)$ the real solution to the equation

$$x(1 + \ln x) = \lambda.$$

   Prove that

$$\lim_{\lambda \to \infty} \frac{f(\lambda)}{\lambda / \ln(\lambda)} = 1.$$

4. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

   for all real numbers x and all positive integers n.

5. For each continuous function $f : [0,1] \to \mathbb{R}$, let $I(f) = \int_0^1 x^2 f(x)\,dx$ and $J(x) = \int_0^1 x\,(f(x))^2\,dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.