2021 - ISU Putnam Practice Set 4

Friday, October 27, 2022

Matrices

1. Do there exist $n \times n$ matrices A and B such that $AB - BA = I_n$?

2. Let Abe the $n \times n$ matrix whose i, j entry is $i + j$ for all $i, j = 1, 2, \ldots, n$. What is the rank of A?

3. Let A and B be $n \times n$ matrices, $n \geq 1$, satisfying $AB - B^2A^2 = I_n$ and $A^3 + B^3 = 0$. Prove that $BA - A^2B^2 = I_n$.

4. Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible?

5. Let V be an infinite set of vectors in $\mathbb{R}^n$ containing n linearly independent vectors. A finite subset $S \subseteq V$ is called crucial if the set $V \setminus S$ contains no n linearly independent vectors, but every set $V \setminus T$, with T a proper subset of S does. Prove there are only finitely many crucial subsets of V .