

# 2021 - ISU Putnam Practice Set 4

Wednesday, September 29, 2021

## More Calculus

1. A function  $f : D \rightarrow \mathbb{R}$  (where  $D$  is an interval) is called convex (you might know this better as concave-up) if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all  $0 < \lambda < 1$  and all  $x, y \in D$ . If  $-f$  is convex, then  $f$  is called concave.

Show that if a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is both convex and concave, then  $f(x) = ax + b$  for some constants  $a, b \in \mathbb{R}$ .

2. Show that

$$\sqrt[3]{3 + \sqrt[3]{3}} + \sqrt[3]{3 - \sqrt[3]{3}} < 2\sqrt[3]{3}.$$

3. Show that

$$(\sin x)^{\sin x} < (\cos x)^{\cos x}$$

for all  $0 < x < \frac{\pi}{4}$ .

4. Let  $k$  be a fixed positive integer. The  $n$ -th derivative of  $\frac{1}{x^k - 1}$  has the form  $\frac{P_n(x)}{(x^k - 1)^{n+1}}$  where  $P_n(x)$  is a polynomial. Find  $P_n(1)$ .

5. Let  $f : [a, b] \rightarrow \mathbb{R}$  be a convex function. Show that  $f$  is continuous on  $(a, b)$ .