2023 - ISU Putnam Practice Set 3

Thursday, September 21, 2023

Continuity

1. Let $f : R \to R$ be a continuous function satisfying $f(x) = f(x^2)$ for all $x \in R$. Prove that f is constant.

2. Let a and b be real numbers in the interval $\left(0, \frac{1}{2}\right)$ and let f be a continuous real-valued function such that
$$f(f(x)) = af(x) + bx, \text{ for all } x \in \mathbb{R}.$$
Prove that $f(0) = 0$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous decreasing function. Prove that the system
$$x = f(y),$$
$$y = f(z),$$
$$z = f(x)$$
has a unique solution.

4. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(0) = 1$ and $f(2x) - f(x) = x$, for all $x \in \mathbb{R}$.

5. Let $f : [-1, 1] \to \mathbb{R}$ be a continuous function such that
   (i) $f(x) = \frac{2-x^2}{2} f\left(\frac{x^2}{2-x^2}\right)$ for every x in $[-1, 1]$,
   (ii) $f(0) = 1$, and
   (iii) $\lim_{x\to 1^-} \frac{f(x)}{\sqrt{1-x}}$ exists and is finite.

   Prove that f is unique, and express $f(x)$ in closed form.