

2021 - ISU Putnam Practice Set 3

Friday, October 7, 2022

Sequences and Series

1. Find

$$\lim_{n \rightarrow \infty} |\sin \pi \sqrt{n^2 + n + 1}|.$$

2. Let $S = \{x_1, x_2, \dots, x_n, \dots\}$ be the set of all positive integers that do not contain the digit 9 in their decimal representation. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{x_n}$$

converges.

3. Is the number

$$\sum_{n=1}^{\infty} \frac{1}{2^{n^2}}$$

rational?

4. Let

$$a_n = \sqrt{1 + \left(1 + \frac{1}{n}\right)^2} + \sqrt{1 + \left(1 - \frac{1}{n}\right)^2},$$

for $n \geq 1$. Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_{20}}$$

is a positive integer.

5. Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that x_0, x_1, \dots are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?