

2021 - ISU Putnam Practice Set 3

Wednesday, September 22, 2021

Calculus 1

1. Let $f : [a, b] \rightarrow [a, b]$ be a continuous function. Show that f has a fixed point; i.e. show that there is a $c \in [a, b]$ with $f(c) = c$.
2. Find all positive real solutions to $2^x = x^2$.
3. Show that not all zeros of the polynomial $P(x) = x^4 - \sqrt{7}x^3 + 4x^2 - \sqrt{22}x + 15$ are real.
4. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying:

$$|f(x) - f(y)| \leq |x - y|^2.$$

for all $x, y \in \mathbb{R}$.

5. Let f be a three times differentiable function (defined on \mathbb{R} and real-valued) such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.