Calculus 1

1. Let \( f : [a, b] \to [a, b] \) be a continuous function. Show that \( f \) has a fixed point; i.e. show that there is a \( c \in [a, b] \) with \( f(c) = c \).

2. Find all positive real solutions to \( 2^x = x^2 \).

3. Show that not all zeros of the polynomial \( P(x) = x^4 - \sqrt{7}x^3 + 4x^2 - \sqrt{22}x + 15 \) are real.

4. Find all functions \( f : \mathbb{R} \to \mathbb{R} \) satisfying:

\[
|f(x) - f(y)| \leq |x - y|^2.
\]

for all \( x, y \in \mathbb{R} \).

5. Let \( f \) be a three times differentiable function (defined on \( \mathbb{R} \) and real-valued) such that \( f \) has at least five distinct real zeros. Prove that \( f + 6f' + 12f'' + 8f''' \) has at least two distinct real zeros.