Differential Equations

1. A not uncommon mistake is to believe that the product rule for derivatives says that \((fg)' = f'g'\). If \(f(x) = e^{x^2}\), determine whether there exists an open interval \((a, b)\) and a nonzero function \(g\) defined on \((a, b)\) such that this wrong product rule is true for \(f\) and \(g\) on \((a, b)\).

2. Let \(f\) and \(g\) be differentiable functions on the real line satisfying the equation

\[(f^2 + g^2)f' + (fg)g' = 0.\]

Prove that \(f\) is bounded.

3. Find all twice-differentiable functions defined on the entire real axis that satisfy \(f'(x)f''(x) = 0\) for all \(x\).

4. Solve the differential equation

\[(x - 1)y'' + (4x - 5)y' + (4x - 6)y = xe^{-2x}.\]

5. Let \(f\) be a twice-differentiable real-valued function satisfying

\[f(x) + f''(x) = -xg(x)f'(x),\]

where \(g(x) \geq 0\) for all real \(x\). Prove that \(|f(x)|\) is bounded.