2021 - ISU Putnam Practice Set 2

Wednesday, September 15, 2021

Determinants and Linear Algebra

1. Show that

$$\begin{pmatrix} (x^2+1)^2 & (xy+1)^2 & (xz+1)^2 \\ (xy+1)^2 & (y^2+1)^2 & (yz+1)^2 \\ (xz+1)^2 & (yz+1)^2 & (z^2+1) \end{pmatrix} = 2(x-y)^2(x-z)^2(y-z)^2.$$

2. Find all numbers in the interval $[-2015, 2015]$ that can be equal to the determinant of an 11×11 matrix with entries equal to 1 or -1.

3. Prove that for any integers $x_1, x_2, \ldots, x_n$ and positive integers $k_1, k_2, \ldots, k_n$, the determant

$$\det \begin{pmatrix} x_1^{k_1} & x_2^{k_1} & \cdots & x_n^{k_1} \\ x_1^{k_2} & x_2^{k_2} & \cdots & x_n^{k_2} \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{k_n} & x_2^{k_n} & \cdots & x_n^{k_n} \end{pmatrix}$$

is divisible by $n!$.

4. Let M be an $n \times n$ complex matrix. Prove that there exist Hermitian matrices A and B such that $M = A + iB$.(A matrix X is called Hermitian if $\overline{X^t} = X$).

5. Let A be the $n \times n$ matrix whose entry in the ith row and the jth column is

$$\frac{1}{\min(i,j)\}}$$

for $1 \leq i, j \leq n$. Compute $\det(A)$.