2021 - ISU Putnam Practice Set 1

Friday, September 23, 2022

Induction

1. Prove that $|\sin(nx)| \leq n|\sin(x)|$ for any real number x and any positive integer n.

2. Prove that the Fibonacci sequence satisfies the identity

$$F_{2n+1} = F_{n+1}^2 + F_n^2, \text{ for } n \geq 0.$$

3. Show that any positive integer can be represented as $\pm 1^2 \pm 2^2 \pm \cdots \pm n^2$ for some positive integer n and some choice of signs.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$f\left(\frac{x_1 + x_2}{2}\right) = \frac{f(x_1) + f(x_2)}{2}$$

for any x_1, x_2. Prove that

$$f\left(\frac{x_1 + x_2 + \cdots x_n}{n}\right) = \frac{f(x_1) + f(x_2) + \cdots f(x_n)}{n}$$

for any $x_1, x_2, \ldots, x_n$.

5. Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions.

   (i) $f(f(n)) = n$, for all integers n;

   (ii) $f(f(n+2)+2) = n$ for all integers n;

   (iii) $f(0) = 1$.