2021 - ISU Putnam Practice Set 1

Wednesday, September 8, 2021

Games of No Chance

1. Choose a positive integer n. At each turn one of the players writes a positive integer that does not exceed n, the rule being that the player cannot write a divisor of a number already written. The player who cannot continue loses. Show that player one has a winning strategy.

2. In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

3. In (a version of) the game of Nim, two players start with a pile of n stones. On each turn, a player removes 1, 2, or 3 stones from the pile. The player to take the last stone wins. For which n does player 1 have a winning strategy?

4. Let k and n be integers with $1 \leq k < n$. Alice and Bob play a game with k pegs in a line of n holes. At the beginning of the game, the pegs occupy the k leftmost holes. A legal move consists of moving a single peg to any vacant hole that is further to the right. The players alternate moves, with Alice playing first. The game ends when the pegs are in the k rightmost holes, so whoever is next to play cannot move, and therefore loses. For what values of n and k does Alice have a winning strategy?

5. An integer n, unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select n in an **odd** number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you **must** guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than $2/3$.