Syzygy Bounds on the Regularity of Ideals

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October 28, 2012
University of Arizona
AMS West Fall Sectional Meetings

Notation

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$R(-d) =$ rank one free module with generator in degree d
(so $R(-d)_i = R_{i-d}$)
$I = (f_1, \ldots, f_t) \subset R$ a homogeneous ideal
(i.e. each f_j is in some R_i)

Graded Free Resolutions

Define a minimal, graded free resolution of R/I as follows:

Map R onto R/I

$$0 \leftarrow R/I \leftarrow R$$

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The kernel is I

$$
\begin{array}{ccccc}
0 \leftarrow R/I \leftarrow R & & \\
& \nwarrow & \\
& & I
\end{array}
$$

Graded Free Resolutions

Define a minimal, graded free resolution of R/I as follows:

Map a free module onto minimal generators of I

$$\begin{array}{ccccccc} 0 & \leftarrow & R/I & \leftarrow & R & & \bigoplus_j R(-j)^{\beta_{1j}} \\ & & & & & \nwarrow & \downarrow \\ & & & & & & I \end{array}$$

Graded Free Resolutions

Define a minimal, graded free resolution of R/I as follows:

Call the composition φ_1

$$\begin{array}{ccccccc} 0 \leftarrow R/I \leftarrow & R & \xleftarrow{\varphi_1} & \bigoplus_j R(-j)^{\beta_{1j}} \\ & \nwarrow & & \downarrow \\ & & & I \end{array}$$

Graded Free Resolutions

Define a minimal, graded free resolution of R/I as follows:

Map a free module onto $\operatorname{Ker}\varphi_1$. Call the composition φ_2.

$$0 \leftarrow R/I \leftarrow R \stackrel{\varphi_1}{\leftarrow} \bigoplus_j R(-j)^{\beta_{1j}} \stackrel{\varphi_2}{\leftarrow} \bigoplus_j R(-j)^{\beta_{2j}}$$

$$\bigoplus_j R(-j)^{\beta_{2j}} \to \operatorname{Ker}\varphi_1 \to \bigoplus_j R(-j)^{\beta_{1j}}$$

Graded Free Resolutions

Define a minimal, graded free resolution of R/I as follows:

Keep Going

$$0 \leftarrow R/I \leftarrow R \stackrel{\varphi_1}{\leftarrow} \bigoplus_j R(-j)^{\beta_{1j}} \stackrel{\varphi_2}{\leftarrow} \bigoplus_j R(-j)^{\beta_{2j}} \stackrel{\varphi_3}{\leftarrow} \cdots$$

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Yields a sequence of graded free modules and graded maps.

$$0 \leftarrow R/I \leftarrow R \stackrel{\varphi_1}{\leftarrow} \bigoplus_j R(-j)^{\beta_{1j}} \stackrel{\varphi_2}{\leftarrow} \bigoplus_j R(-j)^{\beta_{2j}} \stackrel{\varphi_3}{\leftarrow} \cdots \stackrel{\varphi_p}{\leftarrow} \bigoplus_j R(-j)^{\beta_{pj}} \leftarrow 0$$

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Projective Dimension of $R/I = \mathrm{pd}(R/I) = \max\{i \mid \beta_{ij} \neq 0\}$

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Projective Dimension of $R/I = \mathrm{pd}(R/I) = \max\{i \mid \beta_{ij} \neq 0\}$

Regularity of $R/I = \mathrm{reg}(R/I) = \max\{j - i \mid \beta_{ij} \neq 0\}$

Betti Tables

Record the Betti numbers β_{ij} in a matrix called the Betti table:

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	0	1	2	$\cdots$	i	$\cdots$
0:	$\beta_{0,0}$	$\beta_{1,1}$	$\beta_{2,2}$	$\cdots$	$\beta_{i,i}$	$\cdots$
1:	$\beta_{0,1}$	$\beta_{1,2}$	$\beta_{2,3}$	$\cdots$	$\beta_{i,i+1}$	$\cdots$
2:	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	$\cdots$	$\beta_{i,i+2}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	
j:	$\beta_{0,j}$	$\beta_{1,j+1}$	$\beta_{2,j+2}$	$\cdots$	$\beta_{i,i+j}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	

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2:	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	$\cdots$	$\beta_{i,i+2}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	
j:	$\beta_{0,j}$	$\beta_{1,j+1}$	$\beta_{2,j+2}$	$\cdots$	$\beta_{i,i+j}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	

$\mathrm{pd}(R/I) =$ last nonzero column in Betti table.
$\mathrm{reg}(R/I) =$ last nonzero row in Betti table.

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$$R/I \longleftarrow R \xleftarrow{\begin{pmatrix} w^2 & x^2 & wy+xz \end{pmatrix}} R(-2)^3 \xleftarrow{\begin{pmatrix} -x^2 & 0 & -wy-xz & -xy & -y^2 \\ w^2 & -wy-xz & 0 & -wz & z^2 \\ 0 & x^2 & w^2 & wx & wy-xz \end{pmatrix}} R(-4)^5$$

$$\xleftarrow{\begin{pmatrix} y & z & 0 & 0 \\ w & 0 & 0 & z \\ 0 & -x & -y & 0 \\ -x & w & z & -y \\ 0 & 0 & w & x \end{pmatrix}} R(-5)^4 \xleftarrow{\begin{pmatrix} -z \\ y \\ -x \\ w \end{pmatrix}} R(-6) \longleftarrow 0$$

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Betti Table for R/I:

	0	1	2	3	4
0:	1	-	-	-	-
1:	-	3	-	-	-
2:	-	-	5	4	1

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$\mathrm{pd}(R/I) = 4$ $\qquad$ $\mathrm{reg}(R/I) = 2$

Doubly Exponential Regularity Bound

Question

Can one bound $\operatorname{reg}(R/I)$ in terms of the maximal degree of any generator of I and $n =$ number of variables of R?

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Yes:

Theorem (Galligo-'79, Giusti-'84, Caviglia-Sbarra-'05)

Let $I \subset K[x_1, \ldots, x_n] = R$ be an ideal generated in degree $\leq d$. Then

$$\operatorname{reg}(R/I) \leq (2d)^{2^{n-2}}.$$

Doubly Exponential Regularity Bound

	0	1	2	3	4	5	⋯
0:	1	?					
1:	-	?					
2:	-	?					
⋮	-	?					
d-1:	-	?					
d:	-	-					
⋮	-	-					
⋮	-	-					
⋮	-	-					
	-	-					

Doubly Exponential Regularity Bound

	0	1	2	3	4	5	$\cdots$
0:	1	?	?	?	?	?	?
1:	-	?	?	?	?	?	?
2:	-	?	?	?	?	?	?
$\vdots$	-	?	?	?	?	?	?
d-1:	-	?	?	?	?	?	?
d:	-	-	?	?	?	?	?
$\vdots$	-	-	?	?	?	?	?
$\vdots$	-	-	?	?	?	?	?
$(2d)^{2^{n-2}}$:	-	-	?	?	?	?	?
$(2d)^{2^{n-2}}+1$:	-	-	-	-	-	-	-

Mayr-Meyer Ideals

Fix $n \in \mathbb{N}$
$R = K[s_m, f_m, c_{i,m}, b_{i,m} \,|\, 1 \leq i \leq 4, 1 \leq m \leq n]$ (10n variables)
I_n = ideal defined by the $10n - 6$ degree ≤ 4 generators:

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Theorem (Bayer-Stillman-'88)

$$\operatorname{reg}(R/I_n) \geq 2^{2^n}$$

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Moral: Cannot expect any bound on $\mathrm{reg}(R/I)$ in terms of just the maximum of the degrees of the generators and number of variables that is not doubly exponential.

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Mayr-Meyer Ideals $\Rightarrow$ No, if $h = 1$.

We prove the answer is Yes, if $h \geq \left\lceil \frac{n}{2} \right\rceil$, where $n =$ number of variables.

A Regularity Bound in Terms of Half of the Syzygies

Theorem (-)

Let $R = K[x_1, x_2, \ldots, x_n]$ and I be a homogeneous ideal. Set $t_i = t_i(R/I) =$ max degree of an ith syzygy. Set $h = \lceil \frac{n}{2} \rceil$. Then

$$\operatorname{reg}(R/I) \leq \sum_{i=1}^{h} t_i + \frac{\prod_{i=1}^{h} t_i}{(h-1)!}.$$

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Proof uses Boij-Söderberg decomposition of the Betti table of R/I into a positive, rational sum of pure diagrams.

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This bound is in general large, but is much smaller than the doubly exponential bound purely in terms of d and n.

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Remark

Not true in general for modules. Given fixed $t_1 < \ldots < t_h$, there exists a module M over R with $t_i(M) = t_i$ for $i = 1, \ldots, h$ and $t_{h+1}(M)$ arbitrarily large.

A Regularity Bound in Terms of Half of the Syzygies

	0	1	2	$\cdots$	h	$\cdots$	$\cdots$
0:	1	?	?	?	?		
1:	-	?	?	?	?		
$t_1 - 1$:	-	?	?	?	?		
$\vdots$	-	-	?	?	?		
$t_2 - 2$:	-	-	?	?	?		
$\vdots$	-	-	-	?	?		
$t_h - h$:	-	-	-	-	?		
$\vdots$	-	-	-	-	-		
$\vdots$	-	-	-	-	-		
	-	-	-	-	-		

A Regularity Bound in Terms of Half of the Syzygies

	0	1	2	$\cdots$	h	$\cdots$	$\cdots$
0:	1	?	?	?	?	?	?
1:	-	?	?	?	?	?	?
$t_1 - 1$:	-	?	?	?	?	?	?
$\vdots$	-	-	?	?	?	?	?
$t_2 - 2$:	-	-	?	?	?	?	?
$\vdots$	-	-	-	?	?	?	?
$t_h - h$:	-	-	-	-	?	?	?
$\vdots$	-	-	-	-	-	?	?
$\sum t_i + \prod t_i/(h-1)!$:	-	-	-	-	-	?	?
$\sum t_i + \prod t_i/(h-1)! + 1$:	-	-	-	-	-	-	-

An Example

Suppose $I \subset R = K[x_1, \ldots, x_5]$ and we are given

$$t_1(R/I) = 11 \qquad t_2(R/I) = 12 \qquad t_3(R/I) = 13$$

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If further, we are given that I is a three-generated ideal, similar methods yield

$$\operatorname{reg}(R/I) \leq 14.$$

Another Regularity Bound for Ideals

Theorem (-)

Let I be a homogeneous ideal in $R = K[x_1, \ldots, x_n]$. Then

$$t_n \leq \max\{t_i + t_{n-i} \,|\, 1 \leq i \leq n-1\}.$$

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i.e. Last regularity jump in the resolution of R/I cannot be large relative to the previous jumps.

Question

Is

$$t_n \leq \min\{t_i + t_{n-i} \mid 1 \leq i \leq n-1\}?$$

True if $\dim(R/I) \leq 1$ and $\mathrm{depth}(R/I) = 0$ (Eisenbud-Huneke-Ulrich) and if $n \leq 3$ by above.

Boij-Söderberg Theory

Definition

A graded free resolution $F_\bullet$ is *pure* if

$$F_i = R(-d_i)^{\beta_{i,d_i}} \qquad \forall i.$$

(ith syzygies live in unique internal degree d_i for each homological degree i.)

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e.g.

$$R \leftarrow R(-2)^3 \leftarrow R(-4)^5 \leftarrow R(-5)^4 \leftarrow R(-6) \leftarrow 0.$$

is pure with degree sequence $d = (0, 2, 4, 5, 6)$.

Boij-Söderberg Theory

Theorem (Herzog-Kühl '84)

If M is Cohen-Macaulay and has pure resolution with degree sequence $d = (d_0, d_1, \ldots, d_n)$*, then*

$$\beta_{i,d_i} = t \prod_{j \neq i} \frac{1}{|d_j - d_i|} \quad \text{some } t.$$

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Set $\overline{\pi}(d_0, d_1, \ldots, d_n)$ to be the pure Betti table with

$$\overline{\beta}_i(d) = \beta_{i,d_i} = \frac{\prod_{j \neq 0} |d_j - d_0|}{\prod_{j \neq i} |d_j - d_i|}.$$

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e.g. $\overline{\pi}(0, 2, 3) = \begin{pmatrix} 1 & \text{-} & \text{-} \\ \text{-} & 3 & 2 \end{pmatrix}$

Boij-Söderberg Theory

Theorem (Eisenbud-Schreyer '07)

The Betti table of a graded Cohen-Macaulay R-module M is a positive rational sum of pure Betti tables

$$\beta(M) = \sum_{\underline{d} \leq d \leq \overline{d}} q_d \overline{\pi}(d), \qquad q_d \in \mathbb{Q}.$$

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Note: $\mu(M) = \sum_j \beta_{0,j}(M) = \sum_d q_d \overline{\beta}_0(d) = \sum_d q_d$

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Note: $\mu(M) = \sum_j \beta_{0,j}(M) = \sum_d q_d \overline{\beta}_0(d) = \sum_d q_d$
If $M = R/I$, $\sum_d q_d = \mu(R/I) = 1$.

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$\Rightarrow \frac{t_{n-i}}{t_n - t_i} < 1$.

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Suppose $t_n > t_i + t_{n-i}$ for all i.
$\Rightarrow \frac{t_{n-i}}{t_n - t_i} < 1$.
$\Rightarrow \overline{\beta}_n(t_0, t_1, \ldots, t_n) = \frac{\prod_{j \neq 0} t_j}{\prod_{j \neq n} t_n - t_j} = \frac{t_n}{t_n} \frac{t_{n-1}}{t_n - t_1} \cdots \frac{t_1}{t_n - t_{n-1}} < 1$.

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$\Rightarrow \overline{\beta}_n(t_0, t_1, \ldots, t_n) = \frac{\prod_{j \neq 0} t_j}{\prod_{j \neq n} t_n - t_j} = \frac{t_n}{t_n} \frac{t_{n-1}}{t_n - t_1} \cdots \frac{t_1}{t_n - t_{n-1}} < 1$.
Lemma: If $d \leq t = (t_0, \ldots, t_n)$, then $\overline{\beta}_n(d) < \overline{\beta}_n(t)$.

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Suppose $t_n > t_i + t_{n-i}$ for all i.

$\Rightarrow \frac{t_{n-i}}{t_n - t_i} < 1.$

$\Rightarrow \overline{\beta}_n(t_0, t_1, \ldots, t_n) = \frac{\prod_{j\neq 0} t_j}{\prod_{j\neq n} t_n - t_j} = \frac{t_n}{t_n}\frac{t_{n-1}}{t_n - t_1} \cdots \frac{t_1}{t_n - t_{n-1}} < 1.$

Lemma: If $d \leq t = (t_0, \ldots, t_n)$, then $\overline{\beta}_n(d) < \overline{\beta}_n(t)$.

B-S Decomposition $\Rightarrow$

$$1 \leq \beta_{n,t_n}(R/I) = \sum_{\substack{d \leq t \\ d_n = t_n}} q_d \overline{\beta}_n(d) \leq \sum_{\substack{d \leq t \\ d_n = t_n}} q_d \overline{\beta}_n(t)$$

$$< \sum_{\substack{d \leq t \\ d_n = t_n}} q_d \leq \sum_d q_d = 1.$$

$\Rightarrow$ Contradiction!

Thank you!

Reference:

- J McCullough, "A Polynomial Bound on the Regularity of an Ideal in Terms of Half of the Syzygies," arXiv:1112.0058v1 [math.AC], to appear in Math. Res. Ltrs.