SUBADDITIVITY OF SYZYGIES OF IDEALS AND RELATED PROBLEMS

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Decidated to Professor David Eisenbud on the occasion of his seventy-fifth birthday

Abstract. In this paper we survey what is known about the maximal degrees of minimal syzygies of graded ideals over polynomial rings. Subadditivity is one such property that is conjectured to hold for certain classes of rings but fails in general. We discuss bounds on degrees of syzygies and regularity given partial information about the beginning of a free resolution, such as degrees of generators or degrees of first syzygies. We also focus specifically on conditions that guarantee an ideal is quadratic with linear resolution for a fixed number of steps. Finally we collect some old and new open problems on degrees of syzygies.

1. Introduction

Graded free resolutions are highly useful vehicles for computing invariants of ideals and modules. Even when restricting to finite minimal graded free resolutions of graded ideals and modules over a polynomial ring, there are questions regarding the structure of such resolutions that we do not yet understand. The aim of this survey paper is to collect known results on the degrees of syzygies of graded ideals and pose some open questions.

Let $\mathbb{K}$ be a field and let $S = \mathbb{K}[x_1, \ldots, x_n]$ denote a standard graded polynomial ring over $\mathbb{K}$. If M is a finitely generated, graded S-module, let $\beta_{i,j}(M) = \dim_k \operatorname{Tor}_i^S(M, k)_j$ denote the graded Betti numbers of M, and let $\bar{t}_i(M) = \sup\{j \mid \beta_{i,j}(M) \neq 0\}$ denote the ith maximal graded shift of M. Thus $\bar{t}_i(M)$ denotes the maximal degree of an element in a minimal generating set of the ith syzygies of M. The shifts $\bar{t}_i(M)$ are primarily of interest due to their connection with another invariant, the regularity $\operatorname{reg}(M)$ of M; indeed one can take $\operatorname{reg}(M) = \max\{\bar{t}_i(M) - i\}$ as a definition of the regularity of M. The underlying question considered in this paper is the following:

Question 1.1. *Which sequences of integers $(\bar{t}_0, \bar{t}_1, \ldots, \bar{t}_m)$ can be realized as $(\bar{t}_0(S/I), \bar{t}_1(S/I), \ldots, \bar{t}_m(S/I))$ for some graded ideal $I \subseteq S$?*

Note that the more general question in which S/I is replaced by an arbitrary graded module M is not so interesting. One of the main results of

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Boij-Söderberg theory, proved by Eisenbud, Fløystad, and Weyman [23] in characteristic 0 and Eisenbud and Schreyer [26] in all characteristics, shows that every strictly increasing sequence $\bar{t}_0 < \bar{t}_1 < \cdots < \bar{t}_c$ of integers with $c \leq n$ can be realized as the shifts of some graded, Cohen-Macaulay, pure S-module of codimension c. On the other hand, one sees immediately that the same statement is not true for cyclic modules S/I. Indeed, the sequence $(0, 1, 3, 4)$ cannot be realized as the maximal graded shifts of any cyclic module S/I. If it could, then I would be generated by linear forms (since $\bar{t}_1(S/I) = 1$) and yet would have minimal quadratic syzygies, which is impossible; any such S/I would be resolved by a Koszul complex with linear differential maps. This degree sequence $(0, 1, 3, 4)$ can be realized by the resolution of $\operatorname{Coker}(M)$, where M is a generic 2×4 matrix. (See Example 2.1.) This explains restricting our attention to ideals and cyclic modules.

After a section to collect notation and background, there are four main sections to this paper, each dealing with refinements to Question 1.1. In Section 3, we consider effective bounds on $\operatorname{reg}(S/I)$ in terms of some initial segment of the maximal shifts. Thought of another way, we address the question of how much of the resolution of an ideal must be computed to get a reasonable bound on its regularity. In Section 4, we consider the subadditivity property of maximal graded shifts; that is, when $\bar{t}_a(S/I) + \bar{t}_b(S/I) \geq \bar{t}_{a+b}(S/I)$ for all $a, b \geq 1$. It is not hard to find examples where this property fails, but for specific classes of ideals, subadditivity has been proved or conjectured. In Section 5, we consider bounds on maximal graded shifts for arbitrary ideals. In Section 6, we focus specifically on ideals generated by quadrics with linear resolutions for a fixed number of steps. We examine geometric and combinatorial conditions which guarantee resolutions of this form. In the final Section 7, we collect some open questions and pose some new problems that we hope will inspire future work in the area.

2. Background

In this section we fix notation used throughout this paper. Let $\mathbb{K}$ denote a field and let $S = \mathbb{K}[x_1, \ldots, x_n]$ denote a polynomial ring over $\mathbb{K}$. We assume throughout that S is standard graded, i.e. $\deg(x_i) = 1$ for $1 \leq i \leq n$. We write S_i for the $\mathbb{K}$-vector space spanned by all degree i homogeneous polynomials so that $S = \bigoplus_{i \geq 0} S_i$ as $\mathbb{K}$-vector spaces. We write $S(-j)$ for a rank one free module with generator in degree j so that $S(-j)_i = S_{i-j}$. We consider the resolutions of graded ideals $I = (f_1, \ldots, f_m)$ and graded modules M of S. Note that I is graded if it has a set of homogeneous generators. We write $\mathbf{F}_\bullet$ for the minimal graded free resolution of M so that $F_i = \bigoplus_j S(-j)^{\beta_{ij}(M)}$. The numbers $\beta_{ij}(M)$ are the graded Betti numbers of M and can alternatively be computed as $\beta_{ij} = \dim_k \operatorname{Tor}_i^S(M, k)_j$. (We drop the module M from the notation when it is clear from context.) In particular, the graded Betti numbers are isomorphism invariants of M. It is convenient to keep track of the graded Betti numbers in a matrix called the Betti table; by convention, we place $\beta_{i,i+j}(M)$ in position (i, j). One of the advantages of the Betti table is that it allows us to quickly read off the projective dimension and regularity of the module being resolved. More precisely, we may define $\operatorname{reg}(M) = \max\{j - i \mid \beta_{ij}(M) \neq 0\}$ and $\operatorname{pd}(M) = \max\{i \mid \beta_{i,j}(M) \neq 0\}$. Thus $\operatorname{reg}(M)$ is the index of the last nonzero row in the Betti table of M and $\operatorname{pd}(M)$ is the index of the last nonzero column.

If we want to study the structure of minimal, graded free resolutions more closely, we can consider the maximal and minimal graded shifts of M. For each $i \geq 0$, set

$$\overline{t}_i(M) = \sup\{j \mid \beta_{ij}(M) \neq 0\}$$

and

$$\underline{t}_i(M) = \inf\{j \mid \beta_{ij}(M) \neq 0\}.$$

In other words, $\overline{t}_i(M)$ ($\underline{t}_i(M)$) denotes the maximal (resp. minimal) degree of an element in a minimal generating set of the ith syzygy module of M. When $M = S/I$ is a cyclic module, $\overline{t}_1(S/I)$ denotes the maximal degree of a minimal generator of I. A module M is called pure if $\overline{t}_i(M) = \underline{t}_i(M)$ for all $0 \leq i \leq \operatorname{pd}(M)$.

Example 2.1. *Let $S = \mathbb{K}[x_1, \ldots, x_8]$ and let $M = \operatorname{Coker}(\mathbf{A})$, where*

$$\mathbf{A} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \end{pmatrix}.$$

is a generic matrix. M is resolved by a Buchsbaum-Rim complex [21, Appendix A2.6] *with the following Betti table.*

	0	1	2	3
0:	2	4	-	-
1:	-	-	4	2

In particular, M is a pure module with $\underline{t}_0(M) = \overline{t}_0(M) = 0$, $\underline{t}_1(M) = \overline{t}_1(M) = 1$, $\underline{t}_2(M) = \overline{t}_2(M) = 3$, and $\underline{t}_3(M) = \overline{t}_3(M) = 4$. This is the example mentioned in the introduction.

In a minimal graded free resolution, it is clear that the minimal graded shifts are strictly increasing, that is $\underline{t}_{i-1}(S/I) < \underline{t}_i(S/I)$ for all $1 \leq i \leq \operatorname{pd}(S/I)$. The maximal graded shifts are strictly increasing up to $\operatorname{ht}(I)$.

Proposition 2.2 ([50, Proposition 2.2], [4, Lemma 5.1])**.** *Let I be a standard graded ideal in a polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$ over a field $\mathbb{K}$. For all $1 \leq i \leq \operatorname{ht}(I)$, one has*

$$\overline{t}_{i-1}(S/I) < \overline{t}_i(S/I).$$

To see that $\underline{t}_{i-1}(S/I) \geq \underline{t}_i(S/I)$ is possible for $i > \operatorname{ht}(I)$, see Example 3.11. For the remainder of the paper, we primarily focus on the maximal graded shifts $\overline{t}_i(S/I)$ for graded cyclic modules S/I.

3. Effective Bounds on Regularity

In this section we consider bounds on the regularity of ideals in terms of some initial segment of the maximal graded shifts. Fix a graded ideal I and write $d(I)$ for the maximal degree of an element in a minimal generating set of I. We recall that $\operatorname{reg}(S/I) = \max\{\bar{t}_i(S/I) - i\}$ and so in particular $\operatorname{reg}(S/I) \geq d(I) - 1 = \bar{t}_1(S/I) - 1$. A natural question is to what extent $\operatorname{reg}(S/I)$ can exceed $d(I)$. Without referencing the number of variables, no such bound is possible - an ideal I generated by a complete intersection of n quadrics has $d(I) = 2$ and $\operatorname{reg}(S/I) = n$. If we fix the number of variables to be n, then there is a well-known doubly exponential upper bound on regularity, due to Bayer and Mumford in characteristic 0, and later Caviglia and Sbarra in all characteristics.

Theorem 3.1 ([7, Proposition 3.8], [14, Corollary 2.7])**.** *Let I be a graded ideal in $S = \mathbb{K}[x_1, \ldots, x_n]$. Then*

$$\operatorname{reg}(I) \leq (2d(I))^{2^{n-2}}.$$

Recall that $\operatorname{reg}(I) = \operatorname{reg}(S/I) + 1$. For recent improvements to the above bound, see [15, Corollary 2.3]. Examples based on a construction of Mayr and Meyer [48] due to Bayer and Mumford [7] show that the above bound is close to best possible. (See also [8] and [43].) Thus even by referencing the number of variables, the best bound on the regularity of a cyclic module in terms of the first maximal graded shift is doubly exponential. One could hope that by taking more of the resolution into account, one might be able to formulate a tighter bound on the regularity of ideals. The construction of Ullery in the next subsection shows that, if we do not reference the length of the resolution or number of variables, this is not possible. However, if we do take into account the length of the resolution, one can achieve at least a polynomial bound on regularity in terms of the first half of the maximal graded shifts.

Theorem 3.2 ([49, Theorem 4.7])**.** *Let $I \subset S = \mathbb{K}[x_1, ..., x_n]$ be a homogenous ideal. Set $h = \lceil \frac{n}{2} \rceil$. Then*

$$\operatorname{reg}(S/I) \leq \sum_{i=1}^{h} \bar{t}_i(S/I) + \frac{\prod_{i=1}^{h} \bar{t}_i(S/I)}{(h-1)!}.$$

The proof of the previous result follows from a careful analysis of the Boij-Söderberg decomposition of the Betti table of S/I. A similar idea also proves [49, Theorem 4.4] and hints that stronger statements might be true; see Conjecture 7.5. Note that this result requires no hypotheses on the ideal I.

Using different techniques, the author proved that there is a linear bound on $\operatorname{reg}(S/I)$ in terms of the first $\operatorname{pd}(S/I) - \operatorname{codim}(I)$ maximal graded shifts in the resolution.

Theorem 3.3 ([50, Corollary 3.7]). *Let $I \subset S = \mathbb{K}[x_1, \dots, x_n]$ be a graded ideal with $p = \operatorname{pd}(S/I)$ and $c = codim(I)$. Then*

$$\operatorname{reg}(S/I) \leq \max_{1 \leq i \leq p-c} \{\bar{t}_i(S/I) + (p-i)\bar{t}_1(S/I)\} + p.$$

Note that again there are no assumptions on the ideal I. In the Cohen-Macaulay case, the result follows from a result of Eisenbud, Huneke, and Ulrich; see Theorem 5.1. The above result follows by reverse induction on p. Ideals in the next subsection show that the above result cannot be substantially improved; there are quadratic ideals of codimension c with linear resolutions for arbitrarily many steps but whose last $c+1$ steps have arbitrarily large degree. See Example 4.1.

One could also hope for better bounds on regularity under some hypotheses on the ideal I. If S/I is Cohen-Macaulay, we have the following natural bound given by Huneke, Migliore, Nagel, and Ulrich.

Theorem 3.4 ([39, Remark 3.1]). *Let $S = \mathbb{K}[x_1, ..., x_n]$ be a polynomial ring over a field $\mathbb{K}$, and let I be a graded ideal in S of height c such that S/I is Cohen-Macaulay. Then $\operatorname{reg}(S/I) \leq c(d(I) - 1)$, with equality if and only if S/I is a complete intersection generated by c forms of degree $d(I)$. In particular,*

$$\operatorname{reg}(S/I) \leq n(d(I) - 1)$$

for all ideals I with S/I Cohen-Macaulay.

On the other hand, if I is merely prime with fixed $d(I)$, even the first syzygies can have arbitrarily large degree.

Theorem 3.5 ([13, Theorem 6.2]). *Fix a positive integer $s \geq 9$ and field $\mathbb{K}$. There exists a nondegenerate prime ideal P in a polynomial ring S over $\mathbb{K}$ with $d(P) = 6$ and $\bar{t}_1(P) = s$.*

In the following two subsections, we show some of the limits on these sort of regularity bounds by showing to what extent the maximal graded shifts of ideals and cyclic modules mimic those of arbitrary graded modules. The prime ideals in the previous theorem are derived from the following construction of Ullery, which we recast as idealizations.

3.1. **Ullery's Designer Ideals via Idealizations.** We first recall the idealization construction. Fix a ring R and an R-module M. The idealization (sometimes called the Nagata idealization or trivial extension) of R by M is the ring denoted $R \ltimes M$, which is $R \times M$ as an abelian group with multiplication given by $(r, x) \cdot (s, y) = (rs, ry + sx)$ for $r, s \in R$ and $x, y \in M$. Idealizations are commonly used for constructing Gorenstein rings, when M is the canonical module of R, but here we will be interested in the situation that $R = S$ is a polynomial ring and M is a finitely generated graded S-module. The following is an algebraic description of certain designer ideals of Ullery described in [61].

Fix an increasing sequence of integers $2 = d_1 < d_2 < d_3 < \cdots < d_n$ and set $S = \mathbb{K}[x_1, \ldots, x_n]$. As previously mentioned, there is a pure, Cohen-Macaulay module M with maximal (and minimal) graded shifts $\overline{t}_0(M) = 1$ and $\overline{t}_i(M) = d_i$ for $1 \leq i \leq n$. Denote by $\mathbf{A}$ the first differential in the minimal free resolution of M so that $M = \mathrm{Coker}(\mathbf{A})$. Note that by our choice of shifts, $\mathbf{A}$ is a matrix of linear forms. We then consider the standard graded ring $S \ltimes M$. If M has m minimal generators, then we can represent $S \ltimes M$ as a homogeneous quotient of the standard graded polynomial ring $T = S[y_1, \ldots, y_m] = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$. To be precise, let $\mathbf{y}$ denote the row matrix $(y_1, \ldots, y_m)$. Then $S \ltimes M \cong T/I$, where $I = (y_1, \ldots, y_m)^2 + (\mathbf{y} \cdot \mathbf{A})$. If we write $I' = (y_1, \ldots, y_m)^2$ and $I'' = (\mathbf{y} \cdot \mathbf{A})$, then we have a graded short exact sequence of T-modules:

$$0 \to \frac{I}{I'} \to \frac{T}{I'} \to \frac{T}{I} \to 0.$$

The middle term T/I' has free resolution $\mathbf{E}_\bullet$ with the structure of an Eagon-Northcott complex. In particular, it is a linear free resolution after the first step of length m. The first term in the short exact sequence has a free resolution of the form $\mathbf{K}_\bullet(\mathbf{x}; T) \otimes_S \mathbf{F}_\bullet$, where $\mathbf{K}_\bullet(\mathbf{x}; T)$ denotes the Koszul complex on T with respect to $x_1, \ldots, x_n$, and $\mathbf{F}_\bullet$ is the minimal free resolution of $\mathrm{Syz}_1(M)$. In particular, much of the structure of the free resolution of M is passed to the resolution of I/I'. Let $\varphi_\bullet : \mathbf{K}_\bullet(\mathbf{x}; T) \otimes_S \mathbf{F}_\bullet \to \mathbf{E}_\bullet$ be a map of complexes lifting the inclusion $I/I' \to T/I'$. It follows from standard homological arguments that $\mathrm{Cone}(\varphi_\bullet)$ is a T-free resolution of T/I. By analyzing the structure of this resolution one can show that it is in fact minimal. For details we refer the reader to [61].

As a consequence of the preceding discussion, we have the following special case of a result of Ullery:

Theorem 3.6 ([61, Theorem 1.3])**.** *Let M be a graded $S = \mathbb{K}[x_1, \ldots, x_n]$-module minimally generated in degree 0 by m elements with strictly increasing maximal shifts $d_i := \overline{t}_i(M)$. Let I be the defining ideal of $S \ltimes M$ in the polynomial ring $T = S[y_1, \ldots, y_m]$ as above. Then*

$$\overline{t}_i(T/I) = \begin{cases} d_i + 1 & \text{for } 1 \leq i \leq n \\ d_n + i - n + 1 & \text{for } n + 1 \leq i \leq n + m. \end{cases}$$

In particular, for any strictly increasing sequence of integers $2 \leq d_1 < d_2 < \cdots < d_n$, there exists an ideal I in a polynomial ring T with $\overline{t}_i(T/I) = d_i$ for $1 \leq i \leq n$.

Thus the maximal shifts of a graded ideal can realize any increasing sequence of integers (beginning with $d_1 \geq 2$) as an initial segment at the expense of a long linear tail of the corresponding resolution. We illustrate this with an example.

Example 3.7. *Fix integers $p, r \geq 1$. We show how to construct an ideal generated by quadrics which has linear syzygies for p steps and a $(p +$*

1)th syzygy of degree $p+r+3$. Let $S = \mathbb{K}[x_1, \ldots, x_{p+2}]$ and let $M = \operatorname{Ext}_S^{p+2}(S/(x_1, \ldots, x_{p+2})^{r+1}, S)(-p-r-2)$. Then M is a pure, Cohen-Macaulay S-module with maximal shifts $(0, 1, 2, \ldots, p, p+1, p+r+2)$. As M has $m = \binom{p+r+1}{r}$ minimal generators, we set $T = S[y_1, \ldots, y_m]$ and $I = (y_1, \ldots, y_m)^2 + (\mathbf{y} \cdot \mathbf{A})$, where $\mathbf{A}$ is the linear presentation matrix of M. Then $S \ltimes M(-1) \cong T/I$, and $\bar{t}_i(T/I) = i+1$ for $1 \le i \le p+1$ and $\bar{t}_{p+2}(T/I) = p+r+3$.

When $p = 1$ and $r = 3$, the module M has Betti table:

	0	1	2	3
0 :	10	24	15	-
1 :	-	-	-	-
2 :	-	-	-	-
3 :	-	-	-	1

while T/I has Betti table:

	0	1	2	3	4	5	6	$\cdots$	11	12	13
0 :	1	-	-	-	-	-	-		-	-	-
1 :	-	79	585	2220	5403	9150	11178	$\cdots$	174	15	-
2 :	-	-	-	-	-	-	-		-	-	-
3 :	-	-	-	-	-	-	-		-	-	-
4 :	-	-	-	1	10	45	120	$\cdots$	45	10	1

It is easy to see the copy of $\mathbf{K}_\bullet(y_1, \ldots, y_{10}; T)$ in the 4-linear strand.

Remark 3.8. When $M = J$ is an ideal, the construction of the resolution of T/J can also be found in [51], where Peeva and the author constructed counterexamples to the Eisenbud-Goto Conjecture by way of Rees-like algebras. The Rees-like algebra of J is $S[Jt, t^2] \subseteq S[t]$. As $S[Jt, t^2]/(t^2) \cong S \ltimes J$, the graded Betti table of the defining ideal of $S[Jt, t^2]$ is the same as that of I above. (Although a different grading is used there to make $S[It, t^2]$ a positively graded ring.)

We will return to the study of quadratic ideals with linear syzygies in Section 6.

3.2. **Graded Bourbaki Ideals.** We saw in Subsection 3.1 that we could construct ideals whose resolutions shared many properties with a given module. Bourbaki ideals give another way to construct ideal analogues of modules while preserving much of the structure of the free resolution. While Bourbaki ideals exist in a much wider context, we limit our attention to graded Bourbaki ideals over polynomial rings and refer the interested reader to [11, Chapter VII, §4.9, Theorem 6] for the more general result.

Let $S = \mathbb{K}[x_1, \ldots, x_n]$ and let M be a finitely generated, torsionfree S-module. A *Bourbaki sequence* for M is a short exact sequence of the form

$$0 \to F \to M \to I \to 0,$$

where F is a finitely generated free S-module and I is an ideal of S. Bourbaki sequences always exist and in the graded setting we can be a bit more precise.

Theorem 3.9 ([36, Theorem 1.2]). *Let $\mathbb{K}$ be an infinite field and let $S = \mathbb{K}[x_1, \ldots, x_n]$. Let M be a finitely generated, graded, torsionfree S-module generated in degree 0 with $\operatorname{rank}(M) = r$. Then there is a graded Bourbaki sequence of the form:*

$$0 \to S^{r-1} \to M \to I(-m) \to 0,$$

where $m \in \mathbb{Z}$ and I is a graded height two ideal of S.

As a result, we have the following corollary

Corollary 3.10. *Let M be a finitely generated, graded, torsionfree S-module generated in degree 0. Then there exists an integer m and a height two graded ideal I such that*

$$\bar{t}_{i+1}(S/I) = \bar{t}_i(I) = \bar{t}_i(M) + m,$$

for all $i \geq 0$. In particular, for any strictly increasing sequence of integers $d_1 < d_2 < \cdots < d_n$, there exists a graded height two ideal I and an integer m (depending on $d_1, d_2, \ldots, d_n$) such that $\bar{t}_i(S/I) = d_i + m$.

In other words, we can construct ideals with any pattern of maximal shifts up to an added constant. We can also use Bourbaki ideals to construct ideals whose maximal graded shifts are not strictly increasing.

Example 3.11. *Let $S = \mathbb{K}[x_1, x_2, x_3]$ and set $M = S/(x, y, z)(+1) \oplus S/(x^3, y^3)(+3)$ so that $\operatorname{Syz}_1(M)$ is torsionfree and has the following Betti table.*

	0	1	2
0 :	5	3	1
1 :	-	-	-
2 :	-	1	-

The coresponding graded, height two Bourbaki ideal $I \subseteq T$ associated to M has following Betti table.

	0	1	2	3
0 :	1	-	-	-
1 :	-	-	-	-
2 :	-	-	-	-
3 :	-	4	3	1
4 :	-	-	-	-
5 :	-	-	1	-

Note that $\bar{t}_2(T/I) = 7$, while $\bar{t}_3(T/I) = 6$.

4. Subadditivity of Syzygies

Again let $S = \mathbb{K}[x_1, \ldots, x_n]$, and fix a graded S-ideal I. Then I is said to satisfy the *subadditivity* condition if

$$\bar{t}_a(S/I) + \bar{t}_b(S/I) \geq \bar{t}_{a+b}(S/I)$$

for all integers $a, b \geq 1$ with $a + b \leq \operatorname{pd}(S/I)$. This is a natural condition from the perspective of the Koszul homology algebra. Write $H_i(\mathbf{x}, S/I)$ to denote the ith Koszul homology of S/I with respect to $x_1, \ldots, x_n$. Since $\beta_{i,j}(S/I) = \dim_k H_i(\mathbf{x}, S/I)_j$, we can interpret the Betti table of S/I as a bigraded decomposition of the Koszul homology algebra $H_*(\mathbf{x}; S/I)$, with the obvious multiplicative structure coming from the Koszul complex. In particular, if $\bar{t}_{a+b}(S/I) > \bar{t}_a(S/I) + \bar{t}_b(S/I)$, then there is a generator of the Koszul homology algebra in homological degree $a + b$.

If I is generated by a homogeneous regular sequence $f_1, \ldots, f_c$, then it follows easily from the structure of the Koszul complex $\mathbb{K}(f_1, \ldots, f_c; S)$ that S/I satisfies the subadditivity condition [52, Proposition 4.1]. On the other hand, the subadditivity condition fails in general, even for Cohen-Macaulay quotients S/I.

Example 4.1. *This example is a modification of* [25, Example 4.4]*. Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1, x_2, x_3, x_4]$. Consider the ideals $C = (x_1^4, x_2^4, x_3^4, x_4^4)$ and $I = C + (x_1 + x_2 + x_3 + x_4)^4$. As $\ell = x_1 + x_2 + x_3 + x_4$ is a strong Lefschetz element for S/C* [34, Theorem 3.35]*, it follows that the h-vector of S/I is* $(1, 4, 10, 20, 30, 36, 34, 20)$*. Let $L = C : I$. Using the Lefshchetz property, we see that L has no generators in degree ≤ 4, except for those of C. As $\operatorname{reg}(S/C) = 12$, the map $(S/C)_5 \to (S/C)_9$ via multiplication by ℓ^4 is surjective. Thus there is a $\dim_{\mathbb{K}}(S/C)_5 - \dim_{\mathbb{K}}(S/C)_9 = 40 - 20 = 20$ dimensional kernel to this map corresponding to 20 generators of L in degree 5. Consider the graded short exact sequence*

$$0 \to (S/L)(-4) \to S/C \to S/I \to 0.$$

As S/C is resolved by a Koszul complex, $\bar{t}_1(S/C) = 4$. The degree 5 generators of L force $\bar{t}_1(S/L(-4)) = \bar{t}_1(S/L) + 4 \geq 9$. (Actually we get equality here but the inequality is all we need.) It follows from the long exact sequence of Tor *that $\bar{t}_2(S/I) \geq 9$ while $\bar{t}_1(S/I) = 4$. Thus the subadditivity property fails for S/I. The full sequence of maximal graded shifts of S/I is* $(0, 4, 9, 10, 11)$.

To construct examples where the subadditivity property fails later in the resolution, we can replicate copies of the ideal I in new sets of variables, four at a time. Their ideal sum is resolved by the tensor product of the copies of the resolution of S/I. For example, taking 3 copies of S/I and tensoring the corresponding resolutions, we get the following sequence of maximal graded shifts: $(0, 4, \underline{9}, 13, \underline{18}, 22, \underline{27}, 28, 29, 30, 31, 32, 33)$*. The subadditivity property fails at each of the underlined positions.* $(9 \not\leq 4{+}4$*,* $18 \not\leq 4{+}13$*,* $27 \not\leq 13{+}13.)$

Since the subadditivity condition holds for complete intersections but fails for Cohen-Macaulay ideals, It is natural to ask if it holds for Gorenstein ideals. Some positive results are given by Srinivasan and El Khoury in [27]. However, Gorenstein ideals failing the subadditivity condition were constructed by Seceleanu and the author in [52]. More precisely, they proved the following:

Theorem 4.2. [52, Theorem 4.3] *Let $\mathbb{K}$ be an infinite field and $m \geq 2$ an integer. Then there exists a quadratic, Artinian, Gorenstein ideal I in a polynomial ring S over $\mathbb{K}$ such that I has first syzygies in degree $m+2$. In particular, the subadditivity property fails for S/I.*

The ideals in this construction also come from idealizations but of a different sort. The key is to construct a quadratic Artinian $\mathbb{K}$-algebra A whose defining ideal J has arbitrarily large first syzygies and has the superlevel property. A standard graded $\mathbb{K}$-algebra R is called *superlevel* if its canonical module ω_R is linearly presented over R. In this case, it is sufficient to check that the last differential in the resolution of the defining ideal of A is linear. It follows from a result of Mastroeni, Schenck, and Stillman [47, Theorem 3.5] that the idealization $G = A \ltimes \omega_A(-\operatorname{reg}(A) - 1)$ is Gorenstein, Artininian, and quadratic, and while we do not know the full structure of the defining ideal, minimal syzygies of J induce minimal syzygies of the defining ideal of G.

Nevertheless, there are notable classes of ideals where the subadditivity property is expected or even conjectured:

(1) Monomial ideals,
(2) Koszul algebras,
(3) Toric ideals.

If $I \subseteq S$ is a monomial ideal, the Taylor resolution has maximal graded shifts satisfying the subadditivity property, but this resolution is not minimal in general. When one trims this down to a minimal free resolution, it is not clear that the subadditivity property is preserved, although it is expected and partial results are known. The most general result is the following theorem of Herzog and Srinivasan:

Theorem 4.3. ([37, Corollary 4]) *Let $I \subset S = \mathbb{K}[x_1, \ldots, x_n]$ be a monomial ideal. Then*

$$\bar{t}_1(S/I) + \bar{t}_a(S/I) \geq \bar{t}_{a+1}(S/I)$$

for all integers $0 \leq a < \operatorname{pd}(S/I)$.

Note that the monomial ideal hypothesis is necessary as we have previously seen this inequality fails for arbitrary (even Gorenstein) ideals.

Specific classes of monomials have been shown to satisfy the full subadditivity condition, including facet ideals of simplicial forests [28], edge ideals of certain graphs and hypergraphs [9], and monomial ideals with DGA resolutions [41]; see also [1]. The general case remains open.

The subadditivity property of Koszul algebras was studied by Avramov, Conca, and Iyengar [3, 4], where they explicitly conjecture the subadditivity property and extended work of Backelin [5], Kempf [42] and others. Recall that a standard graded ring $R = S/I$ is called *Koszul* if $R/R_+ \cong K$ has a linear free resolution over R, where R_+ denotes the homogeneous maximal ideal; equivalently, $\bar{t}_i^R(K) = i$ for all $i \geq 0$. While subadditivity is still open in general for Koszul algebras, many slightly weaker results on the maximal graded shifts are known.

If I is generated by quadratic monomials, Conca [16] observed that the following inequalities follow from the Taylor resolution of $R = S/I$:

(1) $\bar{t}_i(R) \leq 2i$ for all $i \geq 0$.
(2) If $\bar{t}_i(R) < 2i$ for some i, then $\bar{t}_{i+1}(R) < 2(i+1)$.
(3) $\bar{t}_i(R) < 2i$ if $i > \dim(S) - \dim(R)$.

Therefore, these same properties hold by a deformation argument whenever I has a quadratic Gröbner basis and it is natural to ask if these properties hold for arbitrary Koszul algebras. Kempf [42, Lemma 4] (and also Backelin [6]) proved that (1) above holds for all Koszul algebras. Items (2) and (3) were proved by Avramov, Conca, and Iyengar [3, Main Theorem]. In a later paper, under mild hypotheses, they proved the following improvements.

Theorem 4.4 ([4, Theorem 5.2])**.** *Suppose $R = S/I$ is a Koszul $\mathbb{K}$-algebra with $\operatorname{Char}(\mathbb{K}) = 0$. Let $m = \min\{i \in \mathbb{Z} \mid \bar{t}_i(R) \geq \bar{t}_{i+1}(R)\}$. Then*

(1) *If $\max\{a, b\} \leq m$, then*
$$\bar{t}_{a+b}(R) \leq \max\{\bar{t}_a(R) + \bar{t}_b(R), \bar{t}_{a-1}(R) + \bar{t}_{b-1}(R) + 3\}.$$
(2) *In particular, if $\max\{a, b\} \leq m$ then*
$$\bar{t}_{a+1}(R) \leq \bar{t}_a(R) + 2$$
and
$$\bar{t}_{a+b}(R) \leq \bar{t}_a(R) + \bar{t}_b(R) + 1.$$

Moreover, we may drop the condition $\max\{a, b\} \leq m$ when R is Cohen-Macaulay, since in this case $\operatorname{ht}(I) = m = \operatorname{pd}_S(R)$.

Minimal free resolutions of toric ideals have similar combinatorial descriptions to those of monomial ideals; for example, see [55, Section 67]. It seems natural to study the subadditivity property for toric ideals. This problem seems wide open.

5. General Syzygy Bounds

As noted in the previous section, the subadditivity condition fails for arbitrary ideals; however, there are several slightly weaker bounds on syzygy degrees that hold with greater generality. In their paper [25], Eisenbud, Huneke, and Ulrich studied the regularity of Tor modules and obtained consequences for ideals $I \subset S$ such that $\dim(S/I) \leq 1$. Note that such results instantly extend to ideals I such that S/I has Cohen-Macaulay defect at most 1; one extends to an infinite base field and then kills a general sequence of linear forms to reduce to this case. In particular, the following two weak convexity results hold when S/I is Cohen-Macaulay.

Theorem 5.1 ([25, Corollary 4.1]). *Suppose $I \subset S = \mathbb{K}[x_1, \ldots, x_n]$ is a graded ideal such that $\dim(S/I) - \operatorname{depth}(S/I) \leq 1$. Set $p = \operatorname{pd}_S(S/I)$. Then for any $0 \leq i \leq p$,*

$$\bar{t}_p(S/I) \leq \bar{t}_{p-i}(S/I) + \bar{t}_i(S/I).$$

Theorem 5.2 ([25, Corollary 4.2(a)]). *Suppose $I \subset S = \mathbb{K}[x_1, \ldots, x_n]$ is a graded ideal such that $\dim(S/I) - \operatorname{depth}(S/I) \leq 1$. Set $p = \operatorname{pd}_S(S/I)$. Suppose that $f_1, \ldots, f_c$ is a homogeneous regular sequence in I, where $d_i = \deg(f_i)$. Then*

$$\bar{t}_p(S/I) \leq \bar{t}_{p-c}(S/I) + \sum_{i=1}^{c} d_i.$$

Both of these results follow from a more general result on the regularity of Tor that requires the hypothesis that $\dim(S/I) - \operatorname{depth}(S/I) \leq 1$; however, it is natural to ask if either of the above results holds without the assumption that $\dim(S/I) - \operatorname{depth}(S/i) \leq 1$; see Conjectures 7.5 and 7.6. While this remains open, slightly weaker statements do hold without assumptions on the ideal I. The author used similar techniques as those used for Theorem 3.2 to show that

$$\bar{t}_p(S/I) \leq \max_{1 \leq i \leq p-1} \left\{ \bar{t}_i(S/I) + \bar{t}_{p-i}(S/I) \right\},$$

where $p = \operatorname{pd}(S/I)$ [49, Theorem 4.4]. Shortly thereafter, Herzog and Srinivasan proved the following stronger statement:

Theorem 5.3 ([37, Corollary 3]). *Let $I \subset S = \mathbb{K}[x_1, \ldots, x_n]$ be a graded ideal and set $p = \operatorname{pd}(S/I)$. Then*

$$\bar{t}_p(S/I) \leq \bar{t}_1(S/I) + \bar{t}_{p-1}(S/I).$$

This result follows from a more general statement [37, Proposition 2], which considers the dual complex of the minimal free resolution of I. Similar techniques yields the stronger statement in Theorem 4.3 for monomial ideals; see Subsection 7.2 for potential stronger statements.

6. Quadratic Ideals and Linear Syzygies

Historically, there has been significant interest in conditions on nondegenerate projective varieties that force the resolutions of the vanishing ideals to be as simple as possible. There are many classical theorems guaranteeing that a variety X is defined by quadrics $q_1, \ldots, q_t$ either ideal theoretically ($I_X = (q_1, \ldots, q_t)$), set theoretically ($I_X = \sqrt{(q_1, \ldots, q_t)}$), or scheme-theoretically ($I_X = (q_1, \ldots, q_t)^{\text{sat}}$). See [31], [53], [58], [59]. Green and Lazarsfeld [29] wrote that "one expects that theorems on generation by quadrics will extend to - and be clarified by - analogous statements for higher syzygies." In this section we consider the stronger condition that I_X is generated by quadrics ideal theoretically and has linear resolution for $p-1$ steps. In many geometric situations, it is natural to assume that the corresponding variety is projectively normal, i.e. that S/I_X is a normal ring. Following Green and Lazarsfeld [30], we define property N_p, sometimes called the Green-Lazarsfeld property, as follows. An ideal $I \subseteq S = \mathbb{K}[x_1, \ldots, x_n]$ satisfies property N_p for some integer $p \geq 0$ if S/I is normal and $\beta^S_{i,j}(I) = 0$ for $j \neq i + 2$ and $0 \leq i < p$. A projective variety X (with fixed embedding) satisfies property N_p if its homogeneous vanishing ideal I_X does. Note that properties N_0 and N_1 are what Mumford termed "normal generation" and "normal presentation" respectively in [53]. Assuming X is projectively normal and nondegenerate, the ideal I_X satisfies property N_p if and only if $\bar{t}_i(S/I_X) = i + 1$ for $1 \leq i \leq p - 1$.

We first consider geometric or combinatorial conditions that ensure an ideal satisfies property N_p. The first example of this type is the following result of Green.

Theorem 6.1 ([31, Theorem 4.a.1])**.** *Let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a smooth projective curve of degree d and genus g. If $d \geq 2g + 1 + p$, then I_X satisfies property N_p.*

This theorem was recovered by Green and Lazarsfeld as a result of the following theorem on points in projective space:

Theorem 6.2 ([29, Theorem 1])**.** *Suppose that $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ consists of $2n+1-p$ points in linear general position (no $n + 1$ lying on a hyperplane). Then I_X satisfies property N_p.*

Specifically regarding curves with their canonical embedding, Green's Conjecture predicts that the N_p property is connected with the Clifford index.

Conjecture 6.3 ([31, Conjecture 5.1])**.** *Let $X \subset \mathbb{P}^g_{\mathbb{C}}$ be a smooth curve in its canonical embedding. Then the Clifford index* $\operatorname{Cliff}(X)$ *is equal to the least integer p for which property N_p fails for I_X.*

See [22, Section 9A] or [46, Section 1.8] for a precise definition of the Clifford index. Note that the hyperelliptic case is simple as X is then a rational normal curve with a linear free resolution. Green and Lazarsfeld [31, Appendix] showed that if $\operatorname{Cliff}(X) = p$, then property N_p fails, so the content of the theorem is the reverse implication. Voisin [62, 63] showed that Green's Conjecture holds for general curves. More recently, a shorter proof of the general case, which also applies in characteristic $p \gg 0$ was given by Aprodu, Farkas, Papadima, Raicu, and Weyman [2] via the theory of Koszul Modules, while Green's Conjecture can fail in small characteristics [60].

For related statements regarding higher dimensional varieties, we refer the reader to the survey [20] by Ein and Lazarsfeld.

Of particular interest are the resolutions of Segre and Veronese varieties. We restrict our discussion to the case when $\operatorname{char}(\mathbb{K}) = 0$, as the resolutions can change in certain small characteristics; see [35]. In the case of Veronese embedding, Green proved the following via a Koszul vanishing argument.

Theorem 6.4 ([32, Theorem 2.2]). *Let $V_{d,r}$ denote the defining ideal of the image of $\mathbb{P}^r$ in $\mathbb{P}^{\binom{r+d}{d}-1}$ under the dth Veronese embedding. Then $V_{d,r}$ satisfies property N_d.*

Ottoviani and Paoletti [54] later conjectured that if $d \geq 3$, then property N_{3d-3} should hold while showing that N_{3d-2} failed. When $d = 2$, the ideals $V_{2,r}$ are generated by the 2×2 minors of a symmetric $(r+1)\times(r+1)$ matrix, whose resolutions are described by Józefiak, Pragacz, and Weyman [40] via representation theoretic techniques. It follows that $V_{2,r}$ satisfies property N_5 and fails N_6 for $r \geq 3$, while $V_{2,2}$ has a linear free resolution. (i.e. satisfies property N_p for all p.) See Section 7 for more on this problem, and see the paper [12] by Bruce, Erman, Goldstein, and Yang for an interesting computational approach.

The situation for Segre embeddings is better understood as resolutions, again in characteristic 0, are given by Lascoux [45] and Pragacz and Weyman [56]. Such ideals are generated by the 2×2 minors of a generic matrix. The next result follows from their construction.

Theorem 6.5. *Let I denote the defining ideal of the Segre embedding $\mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \to \mathbb{P}^{ab-1}$ with $a, b \geq 3$. Then I satisfies property N_3 and fails property N_4.*

When $a = 2$ or $b = 2$, it is well known that the resolution of I is linear. For a more detailed treatment of representation theoretic techniques for computing free resolutions, we refer the reader to the book [65] of Weyman. For summaries of related statements on the N_p property, see [57] and [46].

Especially in combinatorial settings it may not be natural to assume normality; we can also generalize the situation to arbitrary degree. Following [24], we say that an ideal $I \subseteq S = \mathbb{K}[x_1, \ldots, x_n]$ satisfies property $N_{d,p}$ if $\beta^S_{i,j}(I) = 0$ for $j \neq d + i$ and $0 \leq i < p$. This I satisfies property N_p if and only if S/I is normal and I satisfies property $N_{2,p}$.

When I is a square-free monomial ideal, we can identify it with its Stanley-Reisner complex Δ_I whose facets correspond to the monomials not in I. In the specific case when I is generated in degree two, we can identify I with a graph whose vertices correspond to the variables and whose edges $\{i, j\}$ correspond to monomial generators $x_i x_j$ of I. We write $I(G)$ for the edge ideal of the graph G and I_Δ for the square-free monomial ideal corresponding to the simplicial complex Δ. We write $I^\vee = I_{\Delta^\vee}$ for the ideal corresponding to the Alexander dual of the squarefree monomial ideal I.

The following result shows that property $N_{2,p}$ for an edge ideal of a graph can be detected combinatorially.

Theorem 6.6 ([24, Theorem 2.1],[66, Corollary 3.7]). *Let G be a graph and let $p \geq 2$. Then the following are equivalent:*

(1) *$I(G)$ satisfies property $N_{2,p}$.*
(2) *$S/I(G)^\vee$ satisfies Serre's property S_p.*
(3) *G has no induced k-cycle for $4 \leq k \leq p+2$.*

The equivalence of items (1) and (2) is a result of Terai and Yanagawa [66]; the equivalence of (1) and (3) is a result of Eisenbud, Green, Hukek, and Popescu [24] and holds when $p = 1$.

Assuming one has such an edge ideal, Dao, Huneke, and Schweig [19] proved the following logarithmic bound on the regularity.

Theorem 6.7 ([19, Theorem 4.1])**.** *Let G be a graph on n vertices such that $I(G)$ satisfies property $N_{2,p}$ for some integer $p \geq 2$. Then*

$$\operatorname{reg}(S/(I(G)) \leq \log_{\frac{p+3}{2}}\left(\frac{n-1}{p}\right) + 2.$$

For some time it was an open question as to whether there was a bound, independent of the number of variables, on the regularity of quadratic monomial ideals satisfying property $N_{2,p}$ [19, p. 8]. The following theorem of Constantinescu, Kahle, and Varbaro, improving earlier work [17], shows that this is not the case.

Theorem 6.8 ([18, Corollary 6.12])**.** *Suppose* $\operatorname{Char}(\mathbb{K}) = 0$ *and fix positive integers p and r. Then there exists a quadratic square-free monomial ideal $I \subseteq S = \mathbb{K}[x_1, \dots, x_{N(p,r)}]$ satisfying property $N_{2,p}$ with*

$$\operatorname{reg}(S/I) = r.$$

These ideals are constructed via an interesting connection between the regularity of certain edge ideals and the virtual projective dimension of hyperbolic Coxeter groups. It is worth noting though that the construction requires a large number of variables. Also, unlike Ullery's construction in Subsection 3.1, the jump in syzygy degrees cannot happen all at once by Theorem 4.3.

Finally, we recall that Avramov, Conca, and Iyengar proved that Koszul ideals satisfying property $N_{2,p}$ also satisfy a regularity bound.

Theorem 6.9 ([4, Theorem 6.1])**.** *Let $I \subseteq S = \mathbb{K}[x_1, \dots, x_n]$ be a graded ideal such that S/I is Koszul and satisfies property $N_{2,p}$ for some $p \geq 1$. Then*

$$\operatorname{reg}(S/I) \leq 2\left\lfloor \frac{n}{p+1} \right\rfloor + 1.$$

We consider potential stronger regularity bounds in the next section.

7. Questions and Conjectures

We end by collecting a number of open questions and conjectures related to the subadditivity type problems.

7.1. **Subadditivity of Syzygies.** First, we recall the main open cases for the subadditivity condition.

Conjecture 7.1 ([4, Conjecture 5.5])**.** *Let S/I be a Koszul algebra. Then S/I satisfies the subadditivity condition.*

While it is not explicitly conjectured in print, the results in [28, 37] strongly indicate that we should expect the subadditivity condition to hold for all monomial ideals. We make this conjecture here.

Conjecture 7.2. *Let $M \subseteq S$ be a monomial ideal. Then S/M satisfies the subadditivity condition.*

Given the combinatorial nature of resolutions of toric ideals, it seems natural to expect that they also satisfy the subadditivity condition. The author knows of no counterexamples to the following conjecture.

Conjecture 7.3. *Let $I \subseteq S$ be a toric ideal (meaning prime and generated by binomials). Then S/I satisfies the subadditivity condition.*

As both monomial ideals and toric ideals have $\mathbb{Z}^m$-gradings for some integer m, we could strengthen both of the previous conjectures to ask about subadditivity of the multi-graded Betti numbers. It would also be interesting to find other classes of ideals where the subadditivity condition holds. We state this formally as a problem.

Open-ended Problem 7.4. *Find other classes of ideals that satisfy the subadditivity condition.*

7.2. **Weak Convexity of Syzygies.** The results of Eisenbud, Huneke, and Ulrich [25] hold for ideals $I \subseteq S$ with Cohen-Macaulay defect at most 1; that is, $\dim(S/I) - \operatorname{depth}(S/I) \leq 1$. The author previously conjectured that several of these results hold in greater generality. We record these here.

Conjecture 7.5 ([49, Question 5.1])**.** *Let $I \subseteq S$ be a graded ideal and let $p = \operatorname{pd}(S/I)$. Then for any integer $0 \leq i \leq p$,*

$$\bar{t}_p(S/I) \leq \bar{t}_i(S/I) + \bar{t}_{p-i}(S/I).$$

This appears to be open even when $p = 4$ and $i = 2$. Note that Theorem 5.3 shows the conjecture holds in the case $i = 1$.

Conjecture 7.6 ([50, Conjecture 1.4])**.** *Let $I \subseteq S$ be a graded ideal and suppose $f_1, \ldots, f_c \in I$ is a homogeneous regular sequence. Set $d_i = \deg(f_i)$ for $1 = 1, \ldots, c$ and $p = \operatorname{pd}(S/I)$. Then*

$$\bar{t}_p(S/I) \leq \bar{t}_{p-c}(S/I) + \sum_{i=1}^{c} d_i.$$

7.3. **Syzygy Bounds on Regularity.** We know that the subadditivity condition fails in general for Cohen-Macaulay ideals; see Example 4.1. More precisely, there are Cohen-Macaulay ideals generated in fixed degree and with arbitrarily large first syzygies. What is not clear is whether resolutions of Cohen-Macaulay ideals can exhibit more extreme behavior beyond the first two steps.

Question 7.7. *Let $I \subseteq S$ be a graded ideal such that S/I is Cohen-Macaulay. Fix an integer $0 \leq i \leq \mathrm{pd}(S/I)$. Does the following inequality hold:*

$$\bar{t}_i(S/I) \leq \max\{i \cdot \bar{t}_1(S/I), \frac{i}{2} \cdot \bar{t}_2(S/I)\}?$$

Ullery's designer ideals show that the Cohen-Macaulay hypothesis cannot be removed from the previous question.

7.4. **Syzygies of Quadratic Ideals.** Recall that property N_d holds for the dth Veronese embedding of $\mathbb{P}^n_{\mathbb{K}}$ and property N_{3d-2} fails. Ottoviani and Paoletti have conjectured that this is sharp.

Conjecture 7.8 ([54])**.** *For integers $n \geq 2$ and $d \geq 3$ and a field $\mathbb{K}$ of characteristic 0, the defining ideal of the dth Veronese embedding of $\mathbb{P}^n_{\mathbb{K}}$ satisfies N_p if and only if $p \leq 3d - 3$.*

When $d = 2$ and $n \geq 3$, it follows from work of Józefiak, Pragacz, and Weyman [40] that property N_5 holds and N_6 fails. When $n = 1$ (and when $d = n = 2$), the corresponding resolution in linear, i.e. property N_p holds for all p. When $n = 2$, the conjecture holds by work of Birkenhake [10] and and Green [31]. When $d = 3$, the conjecture holds by work of Vu [64]. All other cases are open.

Finally, we recall the following question of Constantinescu, Kahle, and Varbaro regarding the regularity of linearly presented quadratic ideals. In the more restrictive Koszul setting, this question was previously posed by Conca [16, Question 2.8].

Question 7.9 ([18, Question 1.1])**.** *Does there exist a family of quadratically generated, linearly presented, graded ideals $I_n \subseteq \mathbb{K}[x_1, \ldots, x_n]$ such that*

$$\lim_{n\to\infty} \frac{\mathrm{reg}(I_n)}{n} > 0?$$

We may replace n with $\mathrm{pd}(I_n)$ as one can mod out by a regular sequence of linear forms to reduce to the above question. One expects that no such families of ideals exist, so we pose the following stronger question:

Question 7.10. *If $I_n \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is a family of quadratic, homogeneous, linearly presented (i.e. satisfying property $N_{2,2}$) ideals, is*

$$\limsup_{n\to\infty} \frac{\mathrm{reg}(I_n)}{\sqrt{n}} < \infty?$$

Clearly a positive answer to Question 7.10 gives a negative answer to Question 7.9. Let's calibrate on some examples.

(1) Let $V_{d,r}$ denote the defining ideal of the dth Veronese embedding of $\mathbb{P}^r$ in $\mathbb{P}^{\binom{r+1}{d}-1}$ in characteristic 0. This ideal satisfies property N_d by Theorem 6.4. One checks that $\operatorname{reg}(V_{d,r}) = r - \lceil \frac{r+1}{d} \rceil + 2$, while $\operatorname{codim}(V_{d,r}) = \binom{r+d}{d} - r - 1$. Setting $d = 2$ and taking an Artinian reduction (so that $n = \binom{r+2}{2} - r - 1$), we see that the above lim sup is a limit with value $\sqrt{2}$ as n approaches ∞. Note also that this example shows that the stronger question asking if families of $N_{2,p}$ ideals satisfy

$$\limsup_{n\to\infty} \frac{\operatorname{reg}(I_n)}{\sqrt[p]{n}} < \infty$$

fails for $p = 3$, since $V_{2,r}$ satisfies property N_3; see above.

(2) If $G_{1,r}$ is the defining ideal of the Grassmannian of lines in $\mathbb{P}^r$, it is known that $\operatorname{reg}(G_{1,r}) = r-3$ [44, Theorem 5.3], while $G_{1,r}$ is Cohen-Macaulay with $\operatorname{codim}(G_{1,r}) = \binom{r-2}{2}$ [38, Corollary 3.2]. Again taking an Artinian reduction and letting r tend to ∞ we get a limit of $\sqrt{2}$.

(3) If $I_n \subseteq \mathbb{K}[x_1, \dots, x_n]$ is a quadratic monomial ideal satisfying property $N_{2,2}$, then by Theorem 6.7, $\operatorname{reg}(I_n) \le \log_{\frac{5}{2}} \binom{n-1}{2} + 2$. It follows that the above lim sup is 0 for all such families of monomial ideals.

(4) Finally, we can construct quadratic, linearly presented ideals I_r with regularity r via the idealization construction in Subsection 3.1. However, the projective dimension of such ideals is $\frac{r^2+3r}{2} + 4$, which is quadratic in r, meaning the above lim sup is always finite.

See also [4, Section 6] and [33] for relevant examples and computations.

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