2026 – ISU Putnam Practice Set 11

Monday, November 16, 2026
AB Workout

1. Suppose $f : \mathbb{Z} \to \mathbb{Z}$ satisfies

$$f(f(n)) = n, \qquad f(f(n+2)+2) = n \qquad (n \in \mathbb{Z}),$$

and $f(0) = 1$. Prove that $f(n) = 1 - n$ for every integer n.

2. A point of $\mathbb{R}^2$ is called rational if both coordinates are rational. What is the largest possible number of rational points on a circle whose center is not rational?

3. Let $S \subseteq \mathbb{R}$ be closed under multiplication, and write $S = T \sqcup U$. Suppose every product of three elements of T belongs to T, and every product of three elements of U belongs to U; repetitions are allowed. Prove that T or U is closed under multiplication.

4. For a partition π of $\{1, 2, \ldots, 9\}$ into nonempty disjoint parts, let $\pi(x)$ denote the size of the part containing x. Given two such partitions π, π', prove that distinct elements x, y satisfy

$$\pi(x) = \pi(y) \qquad \text{and} \qquad \pi'(x) = \pi'(y).$$

5. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies

$$f(x,y) + f(y,z) + f(z,x) = 0 \qquad \text{for all } x, y, z \in \mathbb{R}.$$

Prove that some function $g : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(x,y) = g(x) - g(y) \qquad \text{for all } x, y \in \mathbb{R}.$$