ISU Putnam Practice Set 7
Monday, October 19, 2026

Differential Equations

1. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = f(-x) \qquad \text{for every } x \in \mathbb{R}.$$

Solution: Since f is differentiable and $f'(x) = f(-x)$, its derivative is also differentiable. Differentiating and then applying the original equation at $-x$ gives

$$f''(x) = -f'(-x) = -f(x).$$

Thus $f(x) = A\cos x + B\sin x$ for real constants A, B. The original equation at $x = 0$ requires $f'(0) = f(0)$, so $B = A$. Conversely, if $f(x) = A(\cos x + \sin x)$, then

$$f'(x) = A(\cos x - \sin x) = f(-x).$$

Therefore all solutions are

$$\boxed{f(x) = A(\cos x + \sin x), \qquad A \in \mathbb{R}.}$$

2. For each real number a, solve the initial value problem

$$y' = y^2 - 2xy + x^2 + 1, \qquad y(0) = a,$$

and determine the largest open interval containing 0 on which its solution exists. For which a is the solution defined on all of $\mathbb{R}$?

Solution: The expression on the right is $(y - x)^2 + 1$, suggesting $z(x) = y(x) - x$. Then

$$z' = z^2, \qquad z(0) = a.$$

Multiplying $z' = z \cdot z$ by the integrating factor $\exp(-\int_0^x z(t)\,dt)$ shows that $z(x) = a\exp(\int_0^x z(t)\,dt)$. Thus $a = 0$ forces $z = 0$, while for $a \neq 0$ the function z never vanishes, so

$$\left(\frac{1}{z}\right)' = -1, \qquad \frac{1}{z(x)} = \frac{1}{a} - x.$$

Thus in every case

$$\boxed{y(x) = x + \frac{a}{1 - ax}}.$$

Direct substitution verifies the equation and initial value wherever the formula is defined. When $a \neq 0$, the solution has a pole at $x = 1/a$, so it cannot extend continuously through that point. The maximal intervals containing 0 are therefore

$$\boxed{\begin{cases} (-\infty, 1/a), & a > 0, \\ \mathbb{R}, & a = 0, \\ (1/a, \infty), & a < 0. \end{cases}}$$

The unique solution defined on all of $\mathbb{R}$ corresponds to $a = 0$ and is $y(x) = x$.

3. Let u, v be twice continuously differentiable real-valued functions on $[0,1]$ satisfying

$$u'' = u + u^3, \qquad v'' = v + v^3,$$

with $u(0) = v(0)$ and $u(1) = v(1)$. Prove that $u = v$ on $[0,1]$. In particular, determine all solutions of

$$y'' = y + y^3, \qquad y(0) = y(1) = 0.$$

Solution: Let $w = u - v$. Subtracting the equations and factoring yields

$$w'' = (1 + u^2 + uv + v^2)w.$$

Set $q(x) = 1 + u(x)^2 + u(x)v(x) + v(x)^2$. Since

$$u^2 + uv + v^2 = \left(u + \frac{v}{2}\right)^2 + \frac{3}{4}v^2 \geq 0,$$

we have $q \geq 1$. Multiply the equation for w by w and integrate. Integration by parts, using $w(0) = w(1) = 0$, gives

$$\int_0^1 q(x)w(x)^2\,dx = \int_0^1 w(x)w''(x)\,dx = -\int_0^1 w'(x)^2\,dx.$$

The left side is nonnegative and the right side is nonpositive. They must both be zero. Because $q \geq 1$ and w is continuous, this forces $w = 0$ everywhere, so $u = v$.

The zero function solves the stated boundary value problem. Applying the uniqueness just proved shows that it is the only solution:

$$\boxed{y(x) = 0 \quad (0 \leq x \leq 1).}$$

4. Suppose $a < b$ are real numbers and $y : (a,b) \to \mathbb{R}$ is differentiable with

$$y'(x) \geq 1 + y(x)^2 \qquad (a < x < b).$$

Prove that $b - a \leq \pi$, and determine all such functions when $b - a = \pi$. Deduce that no differentiable real-valued function on all of $\mathbb{R}$ satisfies this differential inequality.

Solution: Set $g(x) = \arctan y(x)$. Then

$$-\frac{\pi}{2} < g(x) < \frac{\pi}{2}, \qquad g'(x) = \frac{y'(x)}{1 + y(x)^2} \geq 1.$$

By the mean value theorem, for $a < s < t < b$,

$$t - s \leq g(t) - g(s) < \pi.$$

Letting $s \to a^+$ and $t \to b^-$ gives $b - a \leq \pi$.

Now suppose $b - a = \pi$ and fix $x \in (a,b)$. For $s < x < t$, the same inequality gives

$$g(x) \geq g(s) + x - s > -\frac{\pi}{2} + x - s,$$

and

$$g(x) \le g(t) - (t - x) < \frac{\pi}{2} - t + x.$$

Letting $s \to a^+$ and $t \to b^-$ yields

$$x - a - \frac{\pi}{2} \le g(x) \le x - b + \frac{\pi}{2}.$$

The two bounds coincide because $b - a = \pi$. Consequently

$$\boxed{y(x) = \tan\left(x - \frac{a+b}{2}\right).}$$

This function is defined throughout (a, b) and satisfies $y' = 1 + y^2$, so it does attain the maximal interval length.

Finally, a solution defined on all of $\mathbb{R}$ could be restricted to an interval of length greater than π, contradicting the bound.

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and $g : \mathbb{R} \to [0, \infty)$ satisfies

$$f''(x) + f(x) = -xg(x)f'(x) \qquad (x \in \mathbb{R}).$$

Show that f is bounded on $\mathbb{R}$.

Solution: (1997 Putnam, Problem B2; wording adapted.) Consider the nonnegative function

$$E(x) = f(x)^2 + f'(x)^2.$$

Using the differential equation, we obtain

$$E'(x) = 2f'(x)\big(f(x) + f''(x)\big) = -2xg(x)f'(x)^2.$$

Because $g \ge 0$, we have $E'(x) \le 0$ for $x \ge 0$ and $E'(x) \ge 0$ for $x \le 0$. Thus E is nonincreasing to the right of 0 and nondecreasing as x approaches 0 from the left. In both cases,

$$E(x) \le E(0) = f(0)^2 + f'(0)^2.$$

Therefore

$$\boxed{|f(x)| \le \sqrt{f(0)^2 + f'(0)^2} \quad \text{for every } x \in \mathbb{R}.}$$

The same argument also bounds $|f'(x)|$. No continuity assumption on g is needed: the derivative sign and the mean value theorem give the required monotonicity.

Source: 1997 Putnam examination, B2; see also Bhargava–Kedlaya–Ng solutions.