ISU Putnam Practice Set 7
Monday, October 19, 2026

Differential Equations

1. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = f(-x) \qquad \text{for every } x \in \mathbb{R}.$$

2. For each real number a, solve the initial value problem

$$y' = y^2 - 2xy + x^2 + 1, \qquad y(0) = a,$$

and determine the largest open interval containing 0 on which its solution exists. For which a is the solution defined on all of $\mathbb{R}$?

3. Let u, v be twice continuously differentiable real-valued functions on $[0, 1]$ satisfying

$$u'' = u + u^3, \qquad v'' = v + v^3,$$

with $u(0) = v(0)$ and $u(1) = v(1)$. Prove that $u = v$ on $[0, 1]$. In particular, determine all solutions of

$$y'' = y + y^3, \qquad y(0) = y(1) = 0.$$

4. Suppose $a < b$ are real numbers and $y : (a, b) \to \mathbb{R}$ is differentiable with

$$y'(x) \geq 1 + y(x)^2 \qquad (a < x < b).$$

Prove that $b - a \leq \pi$, and determine all such functions when $b - a = \pi$. Deduce that no differentiable real-valued function on all of $\mathbb{R}$ satisfies this differential inequality.

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and $g : \mathbb{R} \to [0, \infty)$ satisfies

$$f''(x) + f(x) = -xg(x)f'(x) \qquad (x \in \mathbb{R}).$$

Show that f is bounded on $\mathbb{R}$.