ISU Putnam Practice Set 6
Monday, October 12, 2026

Geometry

1. Two chords AB and CD of a circle of radius R intersect at an interior point P, and the chords are perpendicular. Prove that
$$PA^2 + PB^2 + PC^2 + PD^2 = 4R^2.$$

Solution: Choose coordinates with $P = (0, 0)$, with AB on the horizontal axis and CD on the vertical axis. Write the center of the circle as (h, k). Its equation is

$$(x - h)^2 + (y - k)^2 = R^2.$$

If the horizontal coordinates of A, B are a, b, then they are the roots of

$$x^2 - 2hx + h^2 + k^2 - R^2 = 0.$$

Consequently
$$PA^2 + PB^2 = a^2 + b^2 = (a + b)^2 - 2ab = 2R^2 + 2h^2 - 2k^2.$$

Similarly, applying the same calculation to the vertical coordinates of C, D gives

$$PC^2 + PD^2 = 2R^2 + 2k^2 - 2h^2.$$

Adding proves the identity.

2. Let m_a, m_b, m_c be the lengths of the three medians of a nondegenerate triangle of area K. Prove that these three lengths are the side lengths of a nondegenerate triangle, and determine the area of that triangle in terms of K.

Solution: Use position vectors a, b, c for the vertices, and define the directed median vectors

$$u = \frac{b + c}{2} - a, \qquad v = \frac{c + a}{2} - b, \qquad w = \frac{a + b}{2} - c.$$

Their lengths are m_a, m_b, m_c, and $u + v + w = 0$. Placing these three vectors head to tail therefore forms a closed triangle. We verify that it is nondegenerate and calculate its area.

Translate the original triangle so that $a = 0$, and write $b = x$, $c = y$. Then

$$u = \frac{x + y}{2}, \qquad v = \frac{y - 2x}{2}.$$

For planar vectors r, s, write $\det(r, s)$ for the determinant with columns r, s. Bilinearity gives

$$\det(u, v) = \frac{1}{4}\det(x + y, y - 2x) = \frac{3}{4}\det(x, y).$$

This is nonzero because the original triangle is nondegenerate. The triangle with successive vertices $0, u, u + v$ has the three required side lengths. Its area is

$$\frac{1}{2}|\det(u, u + v)| = \frac{1}{2}|\det(u, v)| = \frac{3}{4}\left(\frac{1}{2}|\det(x, y)|\right) = \boxed{\frac{3}{4}K}.$$

3. Let a, b, c be the side lengths of a nondegenerate triangle with area K. Prove that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}\,K,$$

and determine precisely when equality holds. Hence find the greatest possible area of a triangle satisfying $a^2 + b^2 + c^2 = 12$.

Solution: Place two vertices at $(0, 0)$ and $(c, 0)$, and the third at (x, y), with $y > 0$. The value of x need not lie between 0 and c. Then the other two squared side lengths are $x^2 + y^2$ and $(x-c)^2 + y^2$, while $K = cy/2$. Completing squares gives

$$\begin{aligned} a^2 + b^2 + c^2 - 4\sqrt{3}\,K &= x^2 + y^2 + (x-c)^2 + y^2 + c^2 - 2\sqrt{3}\,cy \\ &= 2\left(x - \frac{c}{2}\right)^2 + 2\left(y - \frac{\sqrt{3}\,c}{2}\right)^2 \\ &\geq 0. \end{aligned}$$

Equality holds exactly when $x = c/2$ and $y = \sqrt{3}\,c/2$, which means the triangle is equilateral.

Under the constraint $a^2 + b^2 + c^2 = 12$, the inequality yields

$$K \leq \frac{12}{4\sqrt{3}} = \sqrt{3}.$$

An equilateral triangle with side length 2 meets the constraint and has area $\sqrt{3}$. Thus the maximum is $\boxed{\sqrt{3}}$, attained exactly by equilateral triangles of side length 2.

4. In the plane, n distinct points are colored red and another n distinct points are colored blue, with no three of the $2n$ points collinear. Prove that the red points can be paired with the blue points so that the n straight line segments joining paired points are pairwise disjoint.

Solution: There are only $n!$ pairings of the red points with the blue points. Choose one that minimizes the sum of the lengths of its joining segments. We claim that its segments are pairwise disjoint.

Suppose segments R_1B_1 and R_2B_2 intersect at X, where the R_i are red and the B_i are blue. All four endpoints are distinct, and the no-three-collinear hypothesis implies that X lies in the interior of both segments. Replace these pairs by R_1B_2 and R_2B_1. The strict triangle inequality gives

$$|R_1B_2| < |R_1X| + |XB_2|, \qquad |R_2B_1| < |R_2X| + |XB_1|.$$

The inequalities are strict because the two original segments lie on distinct lines. Adding, we obtain

$$|R_1B_2| + |R_2B_1| < |R_1B_1| + |R_2B_2|.$$

The new pairs still match each red point with a blue point and leave all other pairs unchanged, contradicting minimality of the total length. Thus no segments intersect, as required.

5. Two squares have areas summing to 1. They must be placed inside a rectangle with their sides parallel to its sides and with disjoint interiors. Determine the smallest constant A such that, regardless of the squares' sizes, some rectangle of area A accommodates both squares.

Solution: (1996 Putnam, Problem A1; wording adapted.) Let the side lengths be $x \geq y > 0$, so $x^2+y^2 = 1$. Disjoint interiors force the squares' horizontal projections or their vertical projections to have disjoint interiors. The rectangle must therefore have one dimension at least $x + y$ and the other at least x. Conversely, placing the squares side by side fits them into an $x \times (x+y)$ rectangle. Thus the least required area is $x(x + y)$.

Write $x = \cos\theta$, $y = \sin\theta$, with $0 < \theta \leq \pi/4$. Then

$$x(x + y) = \frac{1 + \cos(2\theta) + \sin(2\theta)}{2} \leq \frac{1 + \sqrt{2}}{2}.$$

Equality occurs at $\theta = \pi/8$, so the squares with side lengths $\cos(\pi/8)$ and $\sin(\pi/8)$ require this much area. Every other pair fits into a rectangle of no greater area, which can be enlarged to the stated area if necessary. Hence

$$\boxed{A = \frac{1 + \sqrt{2}}{2}}.$$

Source: 1996 Putnam examination, A1; see also Bhargava–Kedlaya solutions.