Generic lines in projective space and the Koszul property

by

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A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY

Major: Mathematics

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The student author, whose presentation of the scholarship herein was approved by the program of study committee, is solely responsible for the content of this dissertation. The Graduate College will ensure this dissertation is globally accessible and will not permit alterations after a degree is conferred.

Iowa State University
Ames, Iowa
2023
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ACKNOWLEDGMENTS

I want to express my heartfelt gratitude to my wife for her unwavering love, support, and encouragement throughout our first decade of marriage. Your patience, understanding, and unwavering faith in me have been a constant source of love, motivation, and joy. You are my best friend, Esther, and I love you.

I also want to thank my advisor Dr. Jason McCullough for his mentorship, guidance, and support. You have been more than an advisor; you have been a mentor and, more importantly, a friend. Your commitment to my education and mathematics has inspired me to become a better mathematician and person.
ABSTRACT

In this thesis, we study the Koszul property of the homogeneous coordinate ring of a generic collection of $m$ lines in $\mathbb{P}^n$ and the homogeneous coordinate ring of a collection of lines in general linear position in $\mathbb{P}^n$. In particular, we show that if $\mathcal{M}$ is a collection of $m$ lines in general linear position in $\mathbb{P}^n$ with $2m \leq n + 1$ and $R$ is the coordinate ring of $\mathcal{M}$, then $R$ is Koszul. Further, if $\mathcal{M}$ is a generic collection of $m$ lines in $\mathbb{P}^n$ and $R$ is the coordinate ring of $\mathcal{M}$ with $m$ even and $m+1 \leq n$ or $m$ is odd and $m+2 \leq n$, then $R$ is Koszul. Lastly, we show if $\mathcal{M}$ is a generic collection of $m$ lines in $\mathbb{P}^n$ such that

$$m > \frac{1}{72} \left(3(n^2 + 10n + 13) + \sqrt{3(n-1)^3(3n+5)}\right),$$

then $R$ is not Koszul. We give a complete characterization of the Koszul property of the coordinate ring of a generic collection of lines in $\mathbb{P}^n$ for $n \leq 6$ or $m \leq 6$. We also determine the Castelnuovo-Mumford regularity of the coordinate ring for generic collections of lines in $\mathbb{P}^n$ and the projective dimension of the coordinate ring of collections of lines in $\mathbb{P}^n$ which are in general linear position.
CHAPTER 1. GENERAL INTRODUCTION

Priddy defined Koszul algebras in 1970; these algebras have been an active area of research in both commutative and noncommutative algebra. A standard graded algebra $R$ over a field $C$ is said to be Koszul if the resolution of $C$ over $R$ is linear; that is, every matrix in the free resolution of $C$ over $R$ has linear forms as entries. At first glance, the Koszul property seems so narrow that only a few algebras would admit this property; this is, in fact, not the case. Indeed, Koszul algebras are ubiquitous in many fields of mathematics. For example, Grassmannians in their Plücker embeddings, coordinate rings of generic sets of projective points, sufficiently high Veronese subrings, quadratic monomial ideals, quadratic complete intersections, and many more are all Koszul.

Furthermore, not only are Koszul algebras ubiquitous, but they possess remarkable numerical and homological properties. For example, a $C$-algebra $R$ is Koszul if and only if

$$H_R(t)P_C^R(-t) = 1.$$ 

This condition tells us that the Poincaré series of $C$ over $R$ must be a rational function, which is not true in general. From a certain perspective, Koszul algebras behave homologically as polynomial rings. For instance, their regularity can be characterized in terms of truncated submodules (Conca et al. (2013)). Another amazing property is that Koszul algebras completely characterize when modules have finite regularity (Avramov and Eisenbud (1992); Avramov and Peeva (2001)). These reasons make Koszul algebras fascinating and loved by commutative algebraists.

The structure of this thesis is the following. Chapter 2 covers preliminary commutative algebra. Chapter 3 gives a brief background on Koszul algebras. It also contains the definition of G-quadratic and LG-quadratic algebras and some interesting numerical properties of Koszul algebras. Finally, in Chapter 4, we present our main results, which concern the Koszul property of a class of algebras.
arising from a generic collection of lines. This chapter was published in Nagoya Mathematical Journal (Rice (2023)).

1.1 References


CHAPTER 2. PRELIMINARIES

In this chapter we review some geometric concepts in projective space and review some of the necessary algebra to understand the results in Chapter 4.

2.1 Projective Space

**Definition 2.1.1.** Given a \(\mathbb{C}^n\)-vector space \(V\) of dimension \(n+1\), the **projective space** \(\mathbb{P}^n\), or \(\mathbb{P}V\), is the set of equivalence classes of \(V - \{0\}\) under the equivalence relation \(\sim\) defined by \(x \sim y\) if and only if there is a non-zero element \(\lambda \in \mathbb{C}\) such that \(x = \lambda y\). A point of \(\mathbb{P}^n\) is denoted as \([x_0 : \cdots : x_n]\), where \(x_i \in \mathbb{C}\) for every \(i\). A **projective variety** \(V\) is a subset of \(\mathbb{P}^n\) that is the zero-locus of a family of homogeneous polynomials in \(n+1\) variables with coefficients in \(\mathbb{C}\). A projective variety is called **irreducible** if it cannot be expressed as a proper union of two projective varieties.

The varieties we will be working with are not irreducible, but they are unions of irreducible varieties. Every projective variety can be studied by passing to an ideal in \(\mathbb{C}[x_0, \ldots, x_n]\) and studying its corresponding quotient ring.

**Definition 2.1.2.** Let \(\mathcal{X}\) be a projective variety in \(\mathbb{P}^n\). Set

\[
\mathcal{I}(\mathcal{X}) = \{ f \in \mathbb{C}[x_0, \ldots, x_n] : f(a_0, \ldots, a_n) = 0 \text{ for all } [a_0 : \cdots : a_n] \in \mathcal{X} \}.
\]

The set \(\mathcal{I}(\mathcal{X})\) is an ideal called the **defining ideal of** \(\mathcal{X}\) and the corresponding quotient ring

\[
R = \mathbb{C}[x_0, \ldots, x_n]/\mathcal{I}(\mathcal{X})
\]

is called the **homogeneous coordinate ring** of \(\mathcal{X}\).

A projective subspace \(\mathbb{P}W\) of \(\mathbb{P}^n\) is of the form \(\pi(W - \{0\})\), where \(\pi\) is the residue class map and \(W\) is a subspace of \(V\). Define \(\dim(\mathbb{P}W) = \dim(W) - 1\). Consider the following example of a projective variety and its homogeneous coordinate ring.
Example 2.1.3. In projective space a line is a subspace of dimension 1. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the following two collections of points in $\mathbb{P}^3$ defined as

$$\mathcal{L}_1 = \{[0 : 0 : \alpha : \beta] : \alpha, \beta \in \mathbb{C} \text{ not both zero}\} \quad \text{and} \quad \mathcal{L}_2 = \{[\alpha : \beta : 0 : 0] : \alpha, \beta \in \mathbb{C} \text{ not both zero}\}.$$ 

Each collection defines a line in $\mathbb{P}^n$, since $\mathcal{L}_1$ is the set of equivalence classes of vectors in the span of $\{e_3, e_4\}$ and $\mathcal{L}_2$ is the set of equivalence classes of vectors in the span of $\{e_1, e_2\}$, where $e_i$ are the standard basis vectors of a 4-dimensional $\mathbb{C}$-vector space. Furthermore, the two defining ideals in $\mathbb{C}[x_0, x_1, x_2, x_3]$ are

$$L_1 = \mathcal{I}(\mathcal{L}_1) = (x_0, x_1) \quad \text{and} \quad L_2 = \mathcal{I}(\mathcal{L}_2) = (x_2, x_3).$$

Thus, the coordinate ring is

$$\mathbb{C}[x_0, x_1, x_2, x_3]/(L_1 \cap L_2) = \mathbb{C}[x_0, x_1, x_2, x_3]/(x_0 x_2, x_0 x_3, x_1 x_2, x_1 x_3).$$

We digress momentarily to define an important geometric concept that we will use heavily.

Definition 2.1.4. Let $\mathcal{P}$ be a collection of $p$ points in $\mathbb{P}^n$ and $\mathcal{M}$ be a collection of $m$ lines in $\mathbb{P}^n$. The points of $\mathcal{P}$ are in general linear position if any $s$ points span a $\mathbb{P}^r$, where $r = \min\{s - 1, n\}$. Similarly, the lines of $\mathcal{M}$ are in general linear position if any $s$ lines span a $\mathbb{P}^r$, where $r = \min\{2s - 1, n\}$. A collection of points in $\mathbb{P}^n$ is a generic collection if every linear form in the defining ideal of each point has algebraically independent coefficients over $\mathbb{Q}$. Similarly, we say a collection of lines is a generic collection if every linear form in the defining ideal of each line has algebraically independent coefficients over $\mathbb{Q}$.

We can interpret this definition as saying generic collections are sufficiently random since a generic collection forms a dense subset of a large parameter space. Furthermore, as one should suspect, a generic collection of lines (points) is in general linear position since collections of lines (points) in general linear position are characterized by the nonvanishing of certain determinants in the coefficients of the defining linear forms and this implication is strict. The previous example demonstrates this fact, and the argument for this example can be found in Example 2.3.5.
For an example of a collection of points in general linear position, simply consider three points in $\mathbb{P}^2$ such that the triple is not collinear (see Example 2.1.5). For an example of lines in general linear position, we only need to consider the collection in Example 2.1.3. This collection of lines must be in general linear position since the pair is skew to one another, meaning that the corresponding planes in $V$ intersect trivially, and so their corresponding projective span must be a 3-dimensional projective subspace. We end this section with examples of points and lines in general linear position demonstrating Definition 2.1.4.

**Example 2.1.5.** Consider $L_1$ and $L_2$ in $\mathbb{P}^3$, from Example 2.1.3. These lines are in general linear position since they span $\mathbb{P}^3$. Now, let $P_1, P_2,$ and $P_3$ be the following three points in $\mathbb{P}^2$:

$$P_1 = [1 : 0 : 0]$$
$$P_2 = [0 : 1 : 0]$$
$$P_3 = [0 : 0 : 1].$$

Each point corresponds to the span of $e_1, e_2,$ and $e_3$ in a 3-dimensional $\mathbb{C}$-vector space $V$, where $e_i$ are the standard basis vectors of $V$. Thus, any pair span a $\mathbb{P}^1$ and all three span $\mathbb{P}^2$. Now, if instead we set

$$P'_3 = [1 : 1 : 0],$$

then the points $P_1, P_2$ and $P'_3$ would not be in general linear position since $P'_3$ corresponds to the span of a vector which lies in the span of $\{e_1, e_2\}$ in $V$; hence, $P'_3$ is on the unique line crossing $P_1$ and $P_2$ in $\mathbb{P}^2$ and the span of all three points is a $\mathbb{P}^1$.

### 2.2 Free Resolutions and Their Betti Numbers

A commutative Noetherian $\mathbb{C}$-algebra $R$ is said to be **graded** if $R = \bigoplus_{i \in \mathbb{N}} R_i$ as a $\mathbb{C}$-vector space such that for all non-negative integers $i$ and $j$, we have $R_i R_j \subseteq R_{i+j}$, and is **standard graded** if $R_0 = \mathbb{C}$ and $R$ is generated as a $\mathbb{C}$-algebra by a finite set of degree 1 elements. Except when explicitly said, all rings are standard graded and Noetherian and all modules are finitely generated. Additionally, an $R$-module $M$ is called graded if $R$ is graded and $M$ can be written
as $M = \bigoplus_{i \in \mathbb{N}} M_i$ as a $\mathbb{C}$-vector space such that for all non-negative integers $i$ and $j$, we have $R_i M_j \subseteq M_{i+j}$. Note each summand $R_i$ and $M_i$ is a $\mathbb{C}$-vector space of finite dimension. Let $S$ be the symmetric algebra of $R_1$ over $\mathbb{C}$; i.e. $S$ is the polynomial ring $S = \mathbb{C}[x_0, \ldots, x_n]$, where $\text{dim}(R_1) = n + 1$ and $x_0, \ldots, x_n$ is a $\mathbb{C}$-basis of $R_1$. We have a surjection $S \to R$ of standard graded $\mathbb{C}$-algebras, and so $R \cong S/J$, where $J$ is a homogeneous ideal of $S$ and the kernel of this map. Denote by $\mathfrak{m}_R$ the maximal homogeneous ideal of $R$. Note, we may view $\mathbb{C}$ as a graded $R$-module since $\mathbb{C} \cong R/\mathfrak{m}_R$.

Free resolution were introduced in Hilbert (1890, 1893) and the key insight was that a free resolution is a description of the structure of $M$. Thus, by studying the resolution of $M$ we can study the properties and structure of $M$. Constructing a resolution amounts to the difficult task of repeatedly solving systems of polynomial equations and is based on the observation that if $A$ is the matrix of the map $R^p \to R^q$ with respect to the fixed bases, then describing the module $\text{Ker}(A)$ is equivalent to solving the system of $R$-linear equations

$$A \begin{pmatrix} Y_1 \\ \vdots \\ Y_p \end{pmatrix} = 0$$

over $R$, where $Y_1, \ldots, Y_p$ are variables that take values in $R$. Many resolutions in this document were computed with the computer program Macaulay2 (Grayson and Stillman).

**Definition 2.2.1.** The minimal graded free resolution $F$ of an $R$-module $M$ is an exact sequence of homomorphisms of finitely generated free graded $R$-modules

$$F : \cdots \to F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \to F_1 \xrightarrow{d_1} F_0,$$

such that $d_{i-1} \circ d_i = 0$ for all $i$, $M \cong F_0 / \text{Im}(d_1)$, and $d_{i+1}(F_{i+1}) \subseteq (x_0, \ldots, x_n) F_i$ for all $i \geq 0$. After choosing bases, we may represent each map in the resolution as a matrix. We can write $F_i = \bigoplus_j R(-j)^{\beta_{ij}^R(M)}$, where $R(-j)$ denotes a rank one free module with a generator in degree $j$, and the numbers $\beta_{ij}^R(M)$ are called the graded Betti numbers of $M$ and are numerical invariants of $M$. The total Betti numbers of $M$ are defined as $\beta_i^R(M) = \sum_j \beta_{ij}^R(M)$. When it is clear
which module we are speaking about, we will write $\beta_{i,j}$ and $\beta_i$ to denote the graded Betti numbers and total Betti numbers, respectively. By construction, we have the equalities

$$\beta^R_i(M) = \dim_\mathbb{C} (\text{Tor}^R_i(M, \mathbb{C})),$$
$$\beta^R_{i,j}(M) = \dim_\mathbb{C} (\text{Tor}^R_i(M, \mathbb{C})_j).$$

Finally, the free resolution $F$ is unique up to isomorphism of complexes.

There is another sense in which a minimal resolution $F$ is minimal. If $G$ is any graded free resolution of an $R$-module $M$, then for some complex $P$, we have an isomorphism $G \cong F \oplus P$, where $P$ is a direct sum of short trivial complexes of the form

$$0 \longrightarrow R(-p) \longrightarrow R(-p) \longrightarrow 0.$$

(Peeva (2011)). Additionally, the condition that $d_{i+1}(F_{i+1}) \subseteq (x_0, \ldots, x_n)F_i$ ensures that no invertible elements appear in the differential matrices. Consider the following example of a minimal graded free resolution.

Example 2.2.2. Let $R = \mathbb{C}[x_0, x_1, x_2, x_3]/(x_0x_2, x_0x_3, x_1x_2, x_1x_3)$, as in Example 2.1.3. The following is the minimal graded free resolution of $R$ over $S$

$$0 \rightarrow S(-4)^1 \rightarrow S(-3)^4 \rightarrow S(-2)^4 \rightarrow S.$$

When constructing a free resolution, there is no reason why one should expect the process to stop. Amazingly, thanks to Hilbert, we know every minimal graded free resolution over $S$ terminates, in the sense that beyond some point within the free resolution, the free modules will be trivial.

Theorem 2.2.3. (Hilbert, 1890, 1893, Hilbert’s Syzygy Theorem) Every finitely generated $S$-module has a free resolution of length at most the number of indeterminates of $S$. 
For proof of Hilbert’s Syzygy Theorem, see Eisenbud (1995). Since Hilbert discovered this fact, significant progress has been made in understanding the structure and properties of finite free resolutions. Much less is known about infinite free resolutions; in fact, most resolutions are infinite.

Infinite resolutions can be difficult to study due to their intricacies and the fact that many of the techniques that worked over $S$ do not work for infinite resolutions. Perhaps, the simplest example of an infinite free resolution can be produced from resolving $C$ over $R = \mathbb{C}[x]/(x^2)$, which yields

\[ \cdots \to R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R, \]

where each map is multiplication by $x$. This example will be a motivating example for our study of Koszul algebras later. We end this section by mentioning that there is a vast amount of literature and open questions asking how the behavior of the Betti numbers of an $R$-module $M$ relate to the structure of the minimal free resolution of $M$, the structure of $M$, and the structure of $R$ that we do not mention here.

### 2.3 Hilbert Functions and Hilbert Series of Homogeneous Coordinate Rings

**Definition 2.3.1.** Let $M$ be a graded $R$-module. The function $\text{Hilb}_M : \mathbb{N} \to \mathbb{N}$ defined by $\text{Hilb}_M(d) = \dim \mathbb{C}(M_d)$ is called the Hilbert function of $M$. Further, there exists a unique polynomial $\text{Hilb}_P(d)$ with rational coefficients, called the Hilbert polynomial, such that $\text{Hilb}_P(d) = \text{Hilb}(d)$ for $d \gg 0$ (Peeva (2011)). The **Hilbert series of $M$** is

\[ H_M(t) = \sum_{i=0}^{\infty} \text{Hilb}_M(i)t^i. \]

The **Poincaré series** of a module $M$ over $R$ is

\[ P^R_M(t) = \sum_{i=0}^{\infty} \beta_i(M)t^i. \]

The **graded Poincaré series** of a module $M$ over $R$ is

\[ Q^R_M(z, t) = \sum_{i=0}^{\infty} \sum_{p \in \mathbb{Z}} \beta_{i,p}(M) z^p t^i. \]
The Hilbert function and Hilbert series encode important information about an $R$-module, such as its dimension and multiplicity (Peeva (2011)). Interestingly, if $M$ is an $S$-module, then the Hilbert series is a rational function. In fact, if we can compute the graded Betti numbers of $M$, then we can compute the Hilbert series, and consequently the Hilbert function since

$$H_M(t) = \frac{\sum_{i,j} (-1)^i \beta_{i,j}^S(M) t^j}{(1 - t)^{n+1}}.$$ 

(2.1)

Once $H_M(t)$ is reduced, so that the numerator and denominator are relatively prime, the numerator is called the $h$-polynomial of $M$. The literature on Hilbert functions and the corresponding Hilbert series is vast and is still being studied, with many open questions (see Peeva and Stillman (2009)).

The Hilbert series of a coordinate ring has other uses other than just carrying numerical information. That is, it is a useful tool to prove various statements. A property that we make significant use of is the fact that the Hilbert series is additive along short exact sequences.

**Proposition 2.3.2.** (Peeva (2011)) Suppose

$$0 \to M' \to M \to M'' \to 0,$$

is a short exact sequence of finitely generated graded $R$-modules. Then,

$$H_M(t) = H_{M'}(t) + H_{M''}(t).$$

Furthermore, if $I$ and $J$ are two ideals of $S$ such that $I \subseteq J$ and $H_{S/I}(t) = H_{S/J}(t)$, then $I = J$.

Determining the Hilbert function of an $R$-module can be incredibly difficult. However, there are certain projective varieties for which we can always compute the Hilbert function of the corresponding coordinate ring.

**Theorem 2.3.3.** (Carlini et al. (2012); Conca et al. (2001)) Let $\mathcal{P}$ be a generic collection of $p$ points in $\mathbb{P}^n$ and $R$ the coordinate ring of $\mathcal{P}$. The Hilbert function of $R$ is

$$\text{Hilb}_R(d) = \min \left\{ \binom{n+d}{d}, p \right\}.$$
In particular, if $p \leq n + 1$, then

$$H_R(t) = \frac{(p - 1)t + 1}{1 - t}.$$ 

The famous Hartshorne-Hirschowitz Theorem gives the Hilbert function in the case of a generic collection of lines.

**Theorem 2.3.4.** *(Hartshorne and Hirschowitz (1982))* Let $\mathcal{M}$ be a generic collection of $m$ lines in $\mathbb{P}^n$ and $R$ the coordinate ring of $\mathcal{M}$. The Hilbert function of $R$ is

$$\text{Hilb}_R(d) = \min\left\{ \binom{n + d}{d}, m(d + 1) \right\}.$$ 

We want to note that this theorem is amazing and incredibly difficult to prove. It is so difficult that Hartshorne’s and Hirschowitz’s proof in the initial case of $\mathbb{P}^3$, which uses degeneration techniques by a smooth quadric surface, occupies more than half the paper. One could ask if any generalization holds for planes in $\mathbb{P}^n$ for $n \geq 5$; unfortunately, this is not known. Moreover, it is not even clear what the correct generalization should be.

We end this section with an example of a coordinate ring of a collection of lines in general linear position that is not generic.

**Example 2.3.5.** Let $\mathcal{L}$ be a collection consisting of the following four lines

$$\mathcal{L}_1 = \{[0 : 0 : \alpha : \beta] : \alpha, \beta \text{ not both zero}\},$$

$$\mathcal{L}_2 = \{[\alpha : \beta : 0 : 0] : \alpha, \beta \text{ not both zero}\},$$

$$\mathcal{L}_3 = \{[\alpha : \beta : -\alpha : \beta] : \alpha, \beta \text{ not both zero}\},$$

$$\mathcal{L}_4 = \{[\alpha : -\beta : \alpha : \beta] : \alpha, \beta \text{ not both zero}\},$$

in $\mathbb{P}^3$. These lines must be in general linear position since every pair is skew and thus spans $\mathbb{P}^3$.

The four defining ideals in $S$ are

$$L_1 = (x_0, x_1), \quad L_2 = (x_2, x_3),$$

$$L_3 = (x_0 + x_2, x_1 - x_3), \quad L_4 = (x_0 - x_2, x_1 + x_3).$$
The defining ideal for $L$ is

$$J = (x_1x_2 + x_0x_3, x_1^3x_3 - x_1x_2^3, x_0x_2^2x_3, x_0x_2x_1x_3, x_0x_2x_3^2, x_0^3x_3 - x_0x_2^2x_3, x_0^3x_2 - x_0x_2^3).$$

Resolving $R = S/J$ over $S$ yields the following free resolution

$$0 \rightarrow S(-6)^3 \xrightarrow{d_3} S(-5)^8 \xrightarrow{d_2} S(-2) \oplus S(-4)^5 \xrightarrow{d_1} S,$$

where

$$d_1 = \left( x_1x_2 + x_0x_3, x_1^3x_3 - x_1x_2^3, x_0x_2^2x_3, x_0x_2x_1x_3, x_0x_2x_3^2, x_0^3x_3 - x_0x_2^2x_3, x_0^3x_2 - x_0x_2^3 \right)$$

and

$$d_2 = \begin{pmatrix}
-x_0^3 + x_0x_2^2 & -x_0x_2x_3 & x_0x_2^3 & 0 & 0 & -x_0^2x_3 & -x_0x_1x_3 & -x_0x_1x_3 + x_0^2x_3 \\
x_1 & 0 & 0 & 0 & -x_3 & 0 & 0 & 0 \\
x_0 & -x_1 & 0 & 0 & x_2 & x_3 & 0 & 0 \\
0 & x_0 & -x_1 & 0 & 0 & x_2 & x_3 & 0 \\
0 & 0 & x_0 & -x_1 & 0 & 0 & x_2 & x_3 \\
0 & 0 & 0 & x_0 & 0 & 0 & 0 & x_2
\end{pmatrix}.$$

Using Equation 2.1 yields the following Hilbert series

$$H_R(t) = \frac{1 - t^2 - 5t^4 + 8t^5 - 3t^6}{(1 - t)^4} = \frac{1 + 2t + 2t^2 + 2t^3 - 3t^4}{(1 - t)^2}.$$  

Now, suppose that $R$ is the coordinate ring for a generic collection of 4 lines in $\mathbb{P}^3$. By Theorem 4.4.4, $R$ would have Hilbert series:

$$H_R(t) = 1 + 4t + 10t^2 + \sum_{i=3}^{\infty} 4(i+1)t^i = \frac{1 + 2t + 3t^2 - 2t^3}{(1 - t)^2}.$$  

So, $L$ is not a generic collection of lines in $\mathbb{P}^3$. 

2.4 Numerical Invariants of a Free Resolution

There are several numerical invariants of a free resolution for a module that can be used to better understand the module.

**Definition 2.4.1.** The **projective dimension** of a finitely generated $R$-module $M$ is

$$
\text{pdim}_R(M) = \sup\{i \mid \beta_i(M) \neq 0\}.
$$

This invariant is quite interesting and measures the length of the resolution of $M$. A consequence of Hilbert’s Syzygy Theorem is that for any finitely generated $S$-module $M$, $\text{pdim}_S(M) \leq n + 1$. The projective dimension of an $R$-module is interesting for another reason; that is, the projective dimension can be related to several other invariants of a module. One particular invariant is the depth of a module.

**Definition 2.4.2.** Let $M$ be an $R$-module. A sequence of elements $r_1, \ldots, r_m$ of $R$ is called an $M$-regular sequence if $r_i$ is a nonzerodivisor on $M/(r_1, \ldots, r_{i-1})M$ for $i = 1, \ldots, m$ and $(r_1, \ldots, r_m)M \neq M$. The depth of $M$, denoted $\text{depth}_R(M)$, is the supremum over all lengths of regular sequences. In the case when $M$ is an $S$-module we write $\text{depth}(M)$. The **Krull dimension**, or dimension, of a module or ring is the supremum of the lengths $k$ of strictly increasing chains $P_0 \subset P_1 \subset \ldots \subset P_k$ of prime ideals of $R$. The dimension of an $R$-module is denoted $\text{dim}(M)$ and is the Krull dimension of the ring $R/I$, where $I = \text{Ann}_R(M)$ is the annihilator of $M$.

The famous Auslander-Buchsbaum formula states that for any finitely generated $R$-module $M$ we have

$$
\text{pdim}_R(M) + \text{depth}_R(M) = \text{depth}(R),
$$

so long as $\text{pdim}_R(M) < \infty$. Letting $R = S$, recovers a precise version of Hilbert’s Syzygy Theorem. In fact, this formula gives us an efficient way of computing bounds on either invariant.

**Proposition 2.4.3.** *(Eisenbud (1995))* Let $R$ be a Noetherian ring and suppose that

$$
0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0
$$

is an exact sequence of finitely generated $R$-modules. Then
(a) \( \text{depth}(M') \geq \min\{\text{depth}(M), \text{depth}(M'') + 1\} \),

(b) \( \text{depth}(M) \geq \min\{\text{depth}(M'), \text{depth}(M'')\} \),

(c) \( \text{depth}(M'') \geq \min\{\text{depth}(M), \text{depth}(M') - 1\} \),

(d) \( \text{dim}(M) = \max\{\text{dim}(M''), \text{dim}(M')\} \).

Furthermore, \( \text{depth}(M) \leq \text{dim}(M) \).

An \( R \)-module \( M \) is Cohen-Macaulay if \( \text{depth}(M) = \text{dim}(M) \). Since \( R \) is a module over itself, we say \( R \) is a Cohen-Macaulay ring if it is a Cohen-Macaulay \( R \)-module. Cohen-Macaulay rings have been studied extensively in commutative algebra, and the definition allows one to study various topics concerning rings. For example, computations in local cohomology are better in the Cohen-Macaulay case and certain dualities behave better in the Cohen-Macaulay case. Furthermore, Cohen-Macaulay rings are ubiquitous; for example, summands of regular rings, complete intersections, and polynomial rings are all Cohen-Macaulay. In summary, Cohen-Macaulay rings are a workhorse in commutative algebra and provide very useful tools and reductions to study rings. Unfortunately, the rings we will study in Chapter 4 are not Cohen-Macaulay, but as we will see, some properties will still be preserved.

Another important invariant, and some could argue the most important, of the free resolution of a graded \( R \)-module \( M \) is its regularity.

**Definition 2.4.4.** Let \( M \) be a finitely generated graded \( R \)-module. The regularity of \( M \) is

\[
\text{reg}(M) = \sup\{j - i \mid \beta_{i,j}^R(M) \neq 0\}.
\]

Similar to the projective dimension of a module, the regularity measures the growth of a free resolution of \( M \). In essence, it measures the complexity of a module. Once more, Hilbert’s Syzygy Theorem shows us that the regularity of every \( S \)-module is finite. A problematic feature of regularity is that, in practice, it tends to be difficult to bound. So another interesting question in commutative algebra is determining if we can obtain sharp upper bounds on the regularity of an \( R \)-module \( M \).
or determine the regularity exactly.

Even though the regularity of a ring can be difficult to work with, not all hope is lost. The regularity behaves nicely over short exact sequences.

**Proposition 2.4.5.** (Eisenbud (1995); Peeva (2011)) Suppose that

\[ 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \]

is an exact sequence of finitely generated graded \( R \)-modules. Then

(a) \( \text{reg}_R(M') \leq \max\{\text{reg}_R(M), \text{reg}_R(M'') + 1\} \),

(b) \( \text{reg}_R(M) \leq \max\{\text{reg}_R(M'), \text{reg}_R(M'')\} \),

(c) \( \text{reg}_R(M'') \leq \max\{\text{reg}_R(M), \text{reg}_R(M') - 1\} \).

Furthermore, if \( d_0 = \min\{d \mid \text{Hilb}(i) = \text{HilbP}(i), \text{for all } i \geq d\} \), then \( \text{reg}_S(M) \geq d_0 \). Moreover, if \( M \) is Cohen-Macaulay, then \( \text{reg}_S(M) = d_0 \). If \( M \) has finite length, then \( \text{reg}_S(M) = \max\{d : M_d \neq 0\} \).

To study these invariants easily, we place the graded Betti numbers of an \( R \)-module \( M \) into a table called the **Betti table**.

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<td>( \beta_{3,3} )</td>
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<td>1</td>
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**Remark 2.4.6.** If an entry is zero in a Betti table, then we write \( - \).

The numerical invariants, mentioned above, of the free resolution of an \( R \)-module can be read off the Betti table: the projective dimension is the width of the table, and the regularity is the height of the table. We close this chapter out with an example.
Example 2.4.7. Consider the free resolution from Example 2.2.2. The Betti table for the $S$-module $R = S/J$ is as follows

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So $\text{pdim}_S(R) = 3$, $\text{reg}_S(R) = 1$, and the $\text{depth}(R) = 1$, by the Auslander Buchsbaum Theorem. Lastly, using the short exact sequence

$$0 \longrightarrow S/J \longrightarrow S/L_1 \oplus S/L_2 \longrightarrow S/(L_1 + L_2) \longrightarrow 0,$$

and Proposition 2.4.3 we get the following equality

\[
2 = \max\{2, 2\} = \max\{\dim(S/L_1), \dim(S/L_2)\} = \dim(S/L_1 \oplus S/L_2) = \max\{\dim(S/J), \dim(S/(L_1 + L_2))\} = \max\{\dim(S/J), 0\} = \dim(S/J).
\]

2.5 References


Grayson, D. R. and Stillman, M. E. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.


CHAPTER 3. INTRODUCTION TO KOSZUL ALGEBRAS

In this chapter, we will provide a very brief introduction to the study of Koszul algebras and their many interesting properties. This chapter is by no means a comprehensive review, and there is a fair amount of literature regarding Koszul algebras and certain stronger notions (Conca et al. (2013); Herzog et al. (2000); Fröberg (1999)).

3.1 Koszul Algebras

It was, and still is, a problem of homological algebra to compute the cohomology algebra of various augmented algebras. The canonical tool for attacking this problem used to be the bar resolution; for example, a classical result states that the cohomology algebra of a Lie algebra $L$ may be computed using the Koszul resolution $U(L) \otimes E(L)$, where $U(L)$ is the universal enveloping algebra of a Lie algebra and $E(L)$ is the exterior algebra of $L$, and this resolution is a subcomplex of the bar resolution. Priddy, motivated by studying the Steenrod Algebra and the universal enveloping algebra, constructed resolutions conceptually analogous to the previous one (Priddy (1970)). Priddy called these resolutions Koszul resolutions and the algebras for which they are defined Koszul algebras.

Since their introduction by Priddy, Koszul algebras have been studied intensely by many different authors under many different names, such as homogeneous preKoszul algebras, Koszul algebras, Fröberg algebras, Priddy rings, wonderful rings, and formal rings. In fact, they have been studied so intensely that in Backelin and Fröberg (1985) there are 18 different equivalent conditions for a ring to be Koszul. Their intense study is because these rings possess extraordinary homological and numerical properties and are connected with many fields of mathematics, such as representation theory, topology, and algebraic geometry. Yet, they are general enough to encompass many classes of rings throughout commutative algebra. For example, any polynomial ring, all quotients
by quadratic monomial ideals, all quadratic complete intersections, the coordinate rings of Grassmannians in their Plücker embedding, and all suitably high Veronese subrings of any standard graded algebra are all Koszul (Conca et al. (2013); Backelin and Fröberg (1985)).

In this chapter, we would like to collect various properties and facts related to Koszul algebras in the commutative setting and illustrate certain implications and counterexamples for various numerical properties that are necessary for a \( C \)-algebra to be Koszul. By no means is this a complete survey of Koszul algebras in the commutative setting, but we will point the reader to sources that give a complete picture. We begin by defining a Koszul algebra.

**Definition 3.1.1.** A standard graded \( C \)-algebra \( R \) is said to be **Koszul** if the \( R \)-module \( C \) has a linear minimal graded free \( R \)-resolution. By linear, we mean each entry in each matrix of the free resolution is a linear form.

The most important feature of the previous definition to notice is that to verify \( R \) is Koszul we must resolve \( C \) over \( R \). Thus, Hilbert’s Syzygy Theorem (Theorem 2.2.3) does not apply, meaning we are usually in the infinite case of free resolutions, so many techniques are unavailable. Furthermore, to add to the difficulty, the Koszul property cannot be determined from computing the first few steps of the resolution of \( C \) (Example 3.1.4).

Consider the following example.

**Example 3.1.2.** Let \( R = \mathbb{C}[x_0]/(x_0^v) \) with \( v \geq 2 \). Then the minimal free resolution of \( C \) over \( R \) is

\[
\cdots \to R(-iv-1) \to R(-iv) \to \cdots \to R(-v) \to R(-1) \to R,
\]

where the maps are multiplication by \( x_0 \) or \( x_0^{v-1} \) depending on the parity. Thus, \( R \) is Koszul if and only if \( v = 2 \). Consequently, \( \text{reg}_R(C) = 0 \) if and only if \( v = 2 \) and \( \text{reg}_R(C) = \infty \) for \( v \geq 3 \). Furthermore, \( \text{pdim}_R(C) = \infty \) for every \( v \geq 2 \).

The previous example illustrates a necessary condition for an algebra to be Koszul. That is, if a \( C \)-algebra \( R = S/J \) is Koszul, then \( J \) is generated by homogeneous forms of degree 2 or 1.

**Theorem 3.1.3.** Let \( R = S/J \) be a Koszul algebra. Then \( J \) is generated by homogeneous forms of degree 2 or 1.
Proof. Suppose that $J$ is minimally generated by $f_1, \ldots, f_m$. We may assume that no $f_i$ is a linear form, since we can reduce to a nondegenerate presentation by quotienting by a regular sequence of linear forms. Consider the following map

$$R(-1)^{n+1} \xrightarrow{d} R,$$

where if $e_i$ is a basis element of $R(-1)^{n+1}$, then $d(e_i) = \bar{x}_i$ for $i = 0, \ldots, n$. We aim to show that $\ker(d)$ is generated by linear elements if and only if $J$ is quadratic. By assumption, we can write $f_i$ as the following

$$f_i = \sum_{j=0}^{n} f_{ij} x_j,$$

for each $i = 1, \ldots, m$ and $f_{ij} \in \mathfrak{m}$. To prove the claim we will show that $\ker(d)$ is minimally generated by

$$u_i = \sum_{j=0}^{n} \bar{f}_{ij} e_j$$

and the Koszul relations

$$r_{ij} = \bar{x}_i e_j - \bar{x}_j e_i,$$

for $i < j$.

Observe that the relations $u_i$ and $r_{ij}$ clearly belong to $\ker(d)$. Let

$$\sum_{j=0}^{n} \bar{g}_j e_j \in \ker(d).$$

It is immediate that $\sum g_j x_j \in J$, so there must exist $h_i \in S$ such that

$$\sum_{j=0}^{n} g_j x_j = \sum_{i=1}^{m} \sum_{i=0}^{n} h_i f_{ij} x_j.$$

Consequently,

$$\sum_{j=0}^{n} \left( g_j - \sum_{i=0}^{n} h_i f_{ij} \right) x_j = 0.$$

This implies that

$$\sum_{j=0}^{n} \left( g_j - \sum_{i=0}^{n} h_i f_{ij} \right) e_j$$
belongs to the kernel of the map $\bigoplus_{j=0}^{n} Se_j \rightarrow (x_0, \ldots, x_n)$ with $e_j \rightarrow x_j$ for $j = 0, \ldots, n$. Since the Koszul complex of $x_0, \ldots, x_n$ over $S$ is acyclic this kernel is generated by the elements

$$s_{kl} = x_k e_l - x_l e_k.$$ 

So, there exist polynomials $p_{kl} \in S$ such that

$$\sum_{j=0}^{n} \left( g_j - \sum_{i=0}^{n} h_i f_{ij} \right) e_j = \sum_{k<l} p_{kl}s_{kl}. $$

Therefore,

$$\sum_{j=0}^{n} \tilde{g}_j e_j = \sum_{i=1}^{m} \tilde{h}_i u_i + \sum_{k<l} \tilde{p}_{kl} r_{kl}. $$

Thus, $u_i$ and $r_{kl}$ generate ker$(d)$.

If this was not a minimal generating set, then we can omit a generator, say $u_1$. Thus, there exists polynomials $q_i$ and $g_{kl}$ in $S$ such that

$$u_1 = \sum_{i=2}^{m} \tilde{q}_i u_i + \sum_{k<l} \tilde{g}_{kl} r_{kl}. $$

Hence,

$$\sum_{j=0}^{n} f_{1j} e_j - \sum_{i=2}^{m} \sum_{j=0}^{n} q_{i} f_{ij} e_j - \sum_{k<l} g_{kl} s_{kl} \in \bigoplus_{j=0}^{n} Je_j. $$

Substituting $x_j$ into the place of $e_j$ we obtain

$$f_1 - \sum_{i=2}^{m} q_{i} f_{i} \in (x_0, \ldots, x_n)J,$$

which is a contradiction to Nakayama’s lemma, since $f_1, \ldots, f_m$ form a minimal system of generators of $J$. 

One might guess that all quadratic algebras are Koszul; unfortunately, more is needed (see Example 3.1.4). Furthermore, even showing a ring admits the Koszul property is usually difficult, but showing a ring is not Koszul is often just as difficult. One could ask if we can determine the Koszul property from computing the first few steps in the resolution of $C$ over $R$. The answer to this question is no; we recall the following example due to Roos.
Example 3.1.4. (Roos (1993)) Let $u$ be a positive natural number such that $u \geq 3$. Define $R_u = \mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5]/I_u$, where $I_u$ is the ideal generated as follows

$$I_u = (x_0^2, x_0x_1, x_1^2, x_1x_2, x_2^2, x_2x_3, x_3^2, x_3x_4, x_4^2, x_4x_5, x_5^2, x_0x_2 + ux_2x_5 - x_3x_5, x_0x_3 + x_2x_5 + (u - 2)x_3x_5).$$

For every $u$, the Hilbert series is $H_{R_u}(t) = 1 + 6t + 8t^2$, which is remarkably independent of $u$. Moreover, the first non-linear syzygy in the resolution of $\mathbb{C}$ over $R_u$ is in homological position $u + 1$. For $u = 6$, the resolution of $\mathbb{C}$ over $R_u$ has the following partial Betti table

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As mentioned previously, and as Roos’s family demonstrates, for an algebra to be Koszul it is not sufficient for it to be quadratic; in fact, the first counterexample of a quadratic algebra that is not Koszul was found by Lech (Example 3.2.4). Thus, a natural question to ask is which quadratic algebras are Koszul. Luckily, we only need to slightly modify Proposition 3.1.3 to recover sufficient conditions for a ring to be Koszul. We delay the proof of the statement until Section 3.3, where we can present a very elegant proof.

Theorem 3.1.5. (Fröberg (1999)) Let $R = S/J$ and $J$ a monomial ideal generated by forms of degree 2. Then $R$ is Koszul.

### 3.2 The Numerics of Koszul Algebras

The numerical properties of Koszul algebras are quite powerful when attempting to prove certain algebras are not Koszul. In this section, we compile a list of numerical properties that Koszul algebras possess and demonstrate how these properties can be used to determine if an algebra is not Koszul. Furthermore, if an algebra $R$ possesses a numerical obstruction $p$ to the Koszul property, then we say $R$ is $p$-obstructed; otherwise, we say $R$ is non-obstructed. We begin with a fundamental characterization of Koszul algebras.
Theorem 3.2.1. (Fröberg (1999)) An algebra $R$ is Koszul if and only if

$$H_R(t)P_C^R(-t) = 1.$$ 

Proof. Let $F$ be the graded minimal free resolution of $C$ over $R$, where each free $R$-module $F_i$ in $F$ has the form $F_i = \bigoplus R(-j)^{\beta_{ij}}$. Hence,

$$1 = \sum_{i \geq 0} \sum_j \beta_{ij} H_R(t)^j (-1)^i = H_R(t)Q_R(-1,t).$$

We aim to show that $R$ is Koszul if and only if

$$P_C^R(-t) = \sum_{i \geq 0} (-1)^i \beta_i t^i = \sum_{i \geq 0} \sum_j \beta_{i,j} (-1)^i t^j = Q_R(-1,t). \tag{3.1}$$

If $R$ is Koszul, then the equality follows immediately. Now, suppose that $R$ is not Koszul and that $\alpha$ is the smallest index for which $\beta_{\alpha,\alpha} \neq \beta_{\alpha}$. The difference

$$\sum_{i \geq 0} \sum_j \beta_{ij} (-1)^i t^j - \sum_{i \geq 0} (-1)^i \beta_{i,i} t^i \neq 0$$

is a formal power series with $(\beta_{\alpha,\alpha} - \beta_{\alpha}) (-1)^\alpha t^\alpha$ as the first non-vanishing term. Therefore, Equation 3.1 does not hold, which proves the claim.

Theorem 3.2.1 is incredibly powerful for determining when an algebra is not Koszul. Additionally, it gives, a necessary condition for a ring to be Koszul; that is, the Maclaurin series expansion of $\frac{1}{H_R(-t)}$ must have non-negative coefficients. In the future, when we invert a Hilbert series and compute its Maclaurin series, we call the coefficients in the expansion $\beta_i'(C)$ the numerical Betti numbers of $C$. Clearly, $\beta_i'(C) = \beta_i^R(C)$ for all $i \geq 0$ if and only if $R$ is Koszul.

The downside of Theorem 3.2.1 is that if one wanted to show an algebra $R$ is Koszul using Theorem 3.2.1, we must know the Hilbert series and the Poincaré series; which, in practice, is quite a lot to know about an algebra. Finally, an interesting feature of the previous theorem is that it shows that the Poincaré series of a Koszul algebra of $C$ is always a rational function, which in general is not always true (see Anick (1982)).

We will present Lech’s example, which is a quadratic algebra, which is not Koszul. Roos’s
example already shows that non-Koszul quadratic algebras exist, but the beauty of Roos’s example is that it is a non-obstructed algebra. So far, Roos’s example has no numerical obstructions to the Koszul property. Before we present Lech’s example of a non-Koszul $H_R(t)$-obstructed algebra, we recall a definition and a powerful theorem.

**Definition 3.2.2.** A homogeneous polynomial is said to be **generic** if its coefficients are algebraically independent over $\mathbb{Q}$.

**Theorem 3.2.3.** *(Hochster and Laksov (1987))* Let $J$ be an ideal minimally generated by $g$ generic forms of degree $d$ in $S = \mathbb{C}[x_1, \ldots, x_n]$. Then

$$\dim_C (J_{d+1}) = \min \left\{ gn, \binom{n+d}{d+1} \right\}.$$  

**Example 3.2.4** *(Lech)*. Let $J$ be an ideal generated by five generic quadrics in $S = \mathbb{C}[x_1, x_2, x_3, x_4]$. By Theorem 3.2.3, the ideal $J$ has the following dimension in degree 3,

$$\dim_C (J_3) = \min \left\{ 20, \binom{6}{3} \right\} = \min \{20, 20\} = 20.$$  

Consequently, we have the following Hilbert series

$$H_{S/J}(t) = 1 + 4t + 5t^2.$$  

Inverting the Hilbert series, evaluating at $-t$, and calculating its Maclaurin series expansion yields a negative coefficient for the $t^6$ term:

$$\frac{1}{H_{S/J}(-t)} = \frac{1}{1 - 4t + 5t^2} = 1 + 4t + 11t^2 + 24t^3 + 41t^4 + 44t^5 - 29t^6 - \cdots .$$

Thus, by Theorem 3.2.1, $S/J$ cannot be Koszul.

A useful feature of Theorem 3.2.1, and the Hilbert series in general, is that it allows us to use theorems from analysis to analyze the behavior of $H_R(t)$ or $\frac{1}{H_R(t)}$ to determine if an algebra admits the Koszul property.

**Theorem 3.2.5.** *(Remmert, 1991, Vivanti–Pringsheim Theorem)* Let the power series $f(z) = \sum_{v=0}^{\infty} a_v z^v$ have positive finite radius of convergence $r$ and suppose that all but finitely many of its coefficients $a_v$ are real and non-negative. Then $z = r$ is a singular point of $f(z)$. 
Remark 3.2.6. Theorems which guarantee a power series has all positive coefficients are rare; furthermore, determining when a power series has all positive coefficients is a difficult problem and is usually handled on an ad-hoc basis. Questions about the positivity of coefficients in power series of reciprocals of polynomials have applications in several fields, such as probability theory, commutative algebra, and combinatorics.

An immediate consequence of Theorem 3.2.5 is a necessary condition for a ring to be Koszul.

Proposition 3.2.7. (Reiner and Welker (2005)) If \( R \) is a Koszul algebra, then its \( h \)-polynomial \( h(t) \) has at least one real root.

Proof. By Theorem 3.2.1, \( P^R_C(t) = \frac{1}{H_R(-t)} \) has non-negative coefficients in its Maclaurin series expansion. So, by Theorem 3.2.5, if \( H_R(t) \) has any real zeros, then \( P^R_C(t) \) will have a pole at \( \rho \), where \( \rho \) is the radius of convergence of \( P^R_C(t) \). Note that \( h(t) \) has a real root exactly when \( h(-t) \) does. So, if \( h(t) \) had no real roots, then \( P^R_C(t) \) has no poles and would have infinite radius of convergence, a contradiction to Theorem 3.2.5.

This proposition is very useful when the \( h \)-polynomial is of low degree. For example, Lech’s example of an algebra defined by 5 generic quadrics has the \( h \)-polynomial \( h(t) = 1 + 4t + 5t^2 \), which has no real root. So, it cannot be Koszul, for two different numerical reasons.

The Koszul property not only places obstructions for the possible Hilbert series of an algebra \( R \), but it can be rather restrictive on the Betti numbers of \( R \), when resolved over \( S \).

Theorem 3.2.8. (Boocher et al. (2017)) Let \( R = S/J \) be a Koszul algebra. If \( J \) is minimally generated by \( g \) elements, then the following hold:

(a) \( \beta^S_i,_{i+1}(R) \leq \binom{i}{2} \) for \( i \in \{2, \ldots, g\} \), and if equality holds for \( i = 2 \), then \( J \) has height one and a linear resolution of length \( g \).

(b) \( \beta^S_i,_{2i}(R) \leq \binom{i}{3} \) for \( i \in \{2, \ldots, g\} \), and if equality holds for \( i = 2 \), then \( J \) is a complete intersection.
Example 3.2.9. Consider Lech’s example (Example 3.2.4). The algebra $R$ has the following (unreduced) Hilbert series

$$H_R(t) = \frac{1 - 5t^2 + 15t^4 - 16t^5 + 5t^6}{(1 - t)^4}.$$

So, the algebra $R$ has the following Betti table resolved over $S$.

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</tbody>
</table>

Thus, this algebra is not Koszul and is $H_R(t)$-obstructed and $\beta^S_{2,4}(R)$-obstructed, since

$$10 = \binom{5}{2} < \beta_{2,4} = 15.$$

Another useful numerical condition for determining if a ring is Koszul, not special to Koszul algebras though, are the deviations of a ring. Specifically, the resolution of $C$ over $R$ is inductively constructed by adjoining $\epsilon_h(R)$ variables at the $h^{th}$ iteration of the Tate construction. The numbers $\epsilon_h(R)$ are called deviations of $R$. As mentioned, the deviations of a ring are not special to Koszul algebras and are especially useful to measure if a ring is a complete intersection. Furthermore, the deviations of a ring act as a measure for how far away a ring is from a complete intersection; if the deviations of a ring are large, then the ring is far from being a complete intersection.

Computing the deviations of a ring, in general, is difficult since we must construct a free resolution of $C$ over $R$. Furthermore, they grow exponentially (Avramov (1999)). Not all hope is lost in the Koszul case though; we need to make a general observation for power series.

Observation 3.2.10. For each formal power series $P(t) = 1 + \sum_{i \geq 1} b_j t^j$, with $b_j \in \mathbb{Z}$, there exists uniquely defined integers $\epsilon'_n$, such that

$$P(t) = \prod_{i=1}^{\infty} \frac{(1 + t^{2i-1})\epsilon'_{2i-1}}{(1 - t^{2i})\epsilon'_{2i}},$$
where the product converges in the \((t)\)-adic topology of \(\mathbb{Z}[[t]]\).

Indeed, let \(p_j(t) = (1 - (-t)^j)^{(-1)^{j+1}}\). Suppose that \(P_0(t) = 1\) and assume by induction that \(P_{n-1} = \prod p_h(t)^{\epsilon'_h}\) satisfies \(P(t) \equiv P_{n-1}(t) \pmod{t^n}\) with uniquely defined \(\epsilon'_h\). If \(P(t) - P_{n-1}(t) \equiv \epsilon_n t^n \pmod{t^{n+1}}\), then set \(P_n(t) = P_{n-1}(t)p_n(t)^{\epsilon'_n}\). The binomial expansion of \(p_n(t)^{\epsilon'_n}\) shows that \(P(t) \equiv P_n(t) \pmod{t^{n+1}}\), and that \(\epsilon'_n\) is the only integer with that property.

**Remark 3.2.11.** The \(b_i\) are defined recursively in terms of \(\epsilon'_n\) as follows

\[
\begin{align*}
    b_1 &= \epsilon'_1 \\
    b_2 &= \epsilon'_2 + \binom{\epsilon'_1}{2} \\
    b_3 &= \epsilon'_3 + \epsilon'_2 \epsilon'_1 + \binom{\epsilon'_1}{3} \\
    b_4 &= \epsilon'_4 + \epsilon'_3 \epsilon'_1 + \binom{\epsilon'_2}{2} + \epsilon'_2 \binom{\epsilon'_1}{2} + \binom{\epsilon'_1}{4}.
\end{align*}
\]

In general, we have the following recursive formula

\[
b_i = \sum \prod_{i_1 + \ldots + i_k = i} \binom{\epsilon'_{i_j}}{h_j},
\]

where \(h_j\) counts the number of times \(i_j\) appears in the partition of \(i\).

Using this observation we can compute numerically what the deviations would be if \(R\) is Koszul. If we expand \(\frac{1}{H_R(-t)}\) as above, we call the numbers \(\epsilon'_h(R)\) the **numerical deviations** of \(R\).

If \(R\) is Koszul, then \(\epsilon'_h(R) = \epsilon_h(R)\) for all \(h\), and hence \(\epsilon'_h \geq 0\). In fact, \(\epsilon'_h(R) > 0\), unless \(R\) is a complete intersection; in this case \(\epsilon_h = 0\) for \(h \geq 3\) (Avramov (1998)). This gives another necessary numerical condition one could use to determine if a ring is Koszul. That is, if \(\epsilon'_h(R) < 0\), for some \(h\), then \(R\) cannot be Koszul.

**Example 3.2.12.** Let \(R\) be as in Lech’s example (Example 3.2.4). Consider the power series

\[
\frac{1}{H_R(-t)} = \frac{1}{1 - 4t + 5t^2} = 1 + 4t + 11t^2 + 24t^3 + 41t^4 + 44t^5 - 29t^6 + \cdots.
\]

We have already determined that \(R\) cannot be Koszul because of the negative coefficient appearing for the \(t^6\) term in the Maclaurin series of \(\frac{1}{H_R(-t)}\). In addition, \(R\) is \(\epsilon'_h\)-obstructed. By Remark 3.2.11, we have the following numerical deviations

\[
\begin{align*}
    \epsilon'_1 &= 4, \\
    \epsilon'_2 &= 11 - \binom{4}{2} = 5, \\
    \epsilon'_3 &= 24 - (5)(4) - \binom{4}{3} = 0.
\end{align*}
\]

Thus, \(R\) is not Koszul (since \(R\) is not a complete intersection).
These necessary numerical conditions give rise to many interesting questions for an algebra $R$; that is, which numerical conditions imply each other? For example, if an algebra $R$ is $H_R(t)$-obstructed, then can it be $\epsilon'_h$-obstructed? If an algebra $R$ is not $\epsilon'_h$-obstructed, then can it be $H_R(t)$-obstructed? We give many counterexamples to these statements in the following example.

**Example 3.2.13.** In the following we present examples of quadratic algebras, none of which are Koszul, to demonstrate that many of necessary numerical conditions for Koszul algebras can fail while others do not. For each of the following statements let $R$ be a $C$-algebra, which is not a complete intersection.

(a) For some $i \geq 0$, $\beta'_i(C) < 0$.

(b) $\beta^S_1(R) = g$ and $\beta^S_{2,4}(R) \geq \binom{g}{2}$.

(c) For some $i \geq 3$, $\epsilon'_i \leq 0$.

(d) For some $j > 4$, $\beta^S_{2,j}(R) \neq 0$.

**Claim 3.2.14.** If an $R$-algebra satisfies (a), then (c) occurs.

**Proof.** By Remark 3.2.11, we have

$$\beta'_i(C) = \sum_{i_1 + \ldots + i_k = i} \prod_{h_j} \binom{\epsilon'_{i, j}}{h_j},$$

and $\epsilon'_h$ are all positive.

**Claim 3.2.15.** There exists a $C$-algebra satisfying (b) but not (d).

**Proof.** See Example 3.2.9.

**Claim 3.2.16.** There exists a $C$-algebra satisfying (d) but not (b).

**Proof.** Let $S = C[x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7]$ and

$$J = (x_0^2, x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2, x_7^2, x_0x_6, x_0x_2 + x_1x_4 + x_3x_7 + x_5x_6).$$
The $\mathbb{C}$-algebra $S/J$ has the following Betti table over $S$.

\[
\begin{array}{cccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
0 & 1 & - & - & - & - & - & - & - & - \\
1 & - & 10 & 2 & - & - & - & - & - & - \\
3 & - & - & 14 & 173 & 200 & 122 & 40 & 5 & - \\
4 & - & - & 5 & 68 & 395 & 632 & 479 & 182 & 28 \\
\end{array}
\]

\[\square\]

**Claim 3.2.17.** There exists a $\mathbb{C}$-algebra $R$ satisfying (a), but failing (b) and (d).

**Proof.** Let $R = S/J$, where $S = \mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5, x_6]$ and $J$ is generated by 15 generic quadrics.

By Theorem 3.2.3,

\[\dim_{\mathbb{C}} (J_3) = \min \left\{ 105, \binom{9}{3} \right\} = \min \left\{ 105, 84 \right\} = 84.\]

Thus,

\[H_R(t) = 1 + 7t + 13t^2.\]

Expanding $\frac{1}{H_R(-t)}$ yields

\[
\frac{1}{H_R(-t)} = 1 + 7t + 36t^2 + 161t^3 + 659t^4 + 2520t^5 + 9073t^6 + 30751t^7 \\
+ 97308t^8 + 281393t^9 + 704747t^{10} + 1275120t^{11} - 235871t^{12} + \cdots,
\]

showing that $R$ satisfies (a). By Remark 3.2.11, the numerical deviations of $R$ are

\[
\epsilon'_1 = 7, \quad \epsilon'_2 = 36 - \binom{7}{2} = 15, \quad \epsilon'_3 = 161 - \binom{15}{1} \binom{7}{1} - \binom{7}{3} = 21,
\]

\[
\epsilon'_4 = 659 - \binom{21}{1} \binom{7}{1} - \binom{15}{2} \binom{7}{2} - \binom{15}{1} \binom{7}{2} - \binom{7}{4} = 57,
\]

\[
\epsilon'_5 = 2520 - \binom{57}{1} \binom{7}{1} - \binom{15}{1} \binom{21}{1} - \binom{7}{2} \binom{21}{1} - \binom{7}{3} \binom{15}{1} - \binom{7}{5} - \binom{15}{2} \binom{7}{1} = 84.
\]
Thus, $R$ satisfies \((c)\).

The algebra $R$ has the following Betti table

\[
\begin{array}{|c|cccccccc|}
\hline
S/J & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
0 & 1 & - & - & - & - & - & - & - \\
1 & - & 15 & 21 & - & - & - & - & - \\
2 & - & - & 63 & 231 & 315 & 225 & 84 & 13 \\
\hline
\end{array}
\]

Therefore, $R$ fails \((b)\) and \((d)\).  

Before we prove the next claim we make use of two theorems and a definition.

**Theorem 3.2.18.** (Stanley (1978)) Let $S = [x_0, \ldots, x_n]$ and $J$ is an ideal generated by $n + 2$ generic polynomials $f_i$ of degree $d_i$. Then

\[
H_{S/J}(t) = \left(\prod_{i=1}^{n+2} \frac{1 - t^{d_i}}{(1 - t)^{a_i}}\right)^{+},
\]

where $(\sum a_i z^i)_+ = \sum b_i z^i$ and $b_i = a_i$, if $a_j \geq 0$ for $j \leq i$ and $b_i = 0$ otherwise.

Some explanation is needed before we present the following definition and theorem. As mentioned in Remark 3.2.6, theorems which guarantee a power series has all positive coefficients are very rare, and proving a power series has all positive coefficients is usually a difficult task. For
example, a conjecture of Lewy and Friedrich, which arose from work on difference approximations to the wave equation, asks if the rational function

\[
\frac{1}{(1-x)(1-y) + (1-y)(1-z) + (1-z)(1-x)}
\]

has all positive coefficients in its Maclaurin series expansion. Szegö settled this conjecture using involved arguments on Bessel functions (Szegö (1933)). This motivated a series of papers using a range of different methods (Askey (1974); Askey and Gasper (1972); Kaluza (1933); Kauers (2007); Kauers and Zeilberger (2008); Pólya (1950)). In fact, the problem of determining whether a power series has all positive coefficients is so difficult that the question of the positivity of the coefficients in the Maclaurin series of

\[
\frac{1}{1 - x - y - z - w + \frac{2}{3}(xy + xz + xw + yz + yw + zw)}
\]

has been wide open since 1972.

The following theorem is very interesting because it gives an algorithm to determine if the Maclaurin series of the reciprocal of an alternating polynomial in a single indeterminate has all positive coefficients. Before we present the theorem, we need a definition from graph theory.

**Definition 3.2.19.** Let $\Gamma$ be a finite gimple graph with vertices $V$ and edges $E$, in which each vertex is assigned a positive integer $j$, called the weight of the vertex $i$. Let $c_{i,j}$ be the number of complete subgraphs of $\Gamma$ with $i$ vertices whose weights sum to $j$. The **vertex weighted clique polynomial** of $\Gamma$ is

\[
c_{\Gamma}(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^i c_{i,j} z^j.
\]

**Theorem 3.2.20.** (Bubenik and Gold (2011)) The associative noncommutative graded algebra

\[
A(\Gamma) = \mathbb{C}\langle x_i \mid i \in V \rangle / I(\Gamma),
\]

where $I(\Gamma) = \langle [x_{i,j}, x_{b_j}] \mid j \in E \rangle$, with $\text{deg}(x_i) = p_i$ has Hilbert series

\[
H_{A(\Gamma)}(t) = \frac{1}{c_{\Gamma}(t)}.
\]
where \( c_{\Gamma}(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^i c_{i,j} z^j \), is the vertex weighted clique polynomial of a vertex weighted graph \( \Gamma \). In particular, the coefficients in the Maclaurin series of \( H_{A(\Gamma)}(t) \) are all positive. Note that \( I(\Gamma) \) is a two-sided ideal.

**Claim 3.2.21.** There exists an \( \mathbb{C} \)-algebra \( R \) failing (a) but satisfying (b), (c), and (d).

**Proof.** Let \( R = S/J \) be a \( \mathbb{C} \)-algebra where \( S = \mathbb{C} = [x_0, \ldots, x_5] \) and \( J = (q_1, \ldots, q_7) \), where \( q_i \) is a generic quadrics. To prove the claim we make use of the following two results. By Theorem 3.2.18, we have the following Hilbert series

\[
H_R(t) = \left( \prod_{i=1}^{7} \frac{(1 - t^2)}{(1 - t)^6} \right) + \\
= (1 + 6t + 14t^2 + 14t^3 - 14t^5 - 14t^6 - 6t^7 - t^8) + \\
= 1 + 6t + 14t^2 + 14t^3 \\
= \frac{1 - 7t^2 + 21t^4 + 14t^5 - 105t^6 + 132t^7 - 70t^8 + 14t^9}{(1 - t)^6}.
\]

Now, consider the following graph \( \Gamma \)

\[
\begin{align*}
&v_7,5 \quad v_8,5 \quad v_9,5 \quad v_{10,5} \quad v_{11,5} \quad v_{12,5} \quad v_{13,5} \quad v_{14,5} \quad v_{15,5} \quad v_{16,5} \quad v_{17,5} \quad v_{18,5} \quad v_{19,5} \quad v_{20,5} \\
&v_6,1 \\
&v_{21,2} \quad v_{22,7} \quad v_{23,7} \quad v_{24,7} \quad v_{25,7} \quad v_{26,7}
\end{align*}
\]

where the index \( j \) on the vertex \( v_{i,j} \) is the weight of the vertex and \( i \) counts the number of vertices.

We claim the vertex weighted clique polynomial of \( \Gamma \) is

\[
c_{\Gamma}(t) = 1 - 6t + 14t^2 - 14t^3.
\]
First consider the subgraph $H$ of $\Gamma$

The graph $H$ is a complete graph on 7 vertices with one vertex weighted 2, and every other vertex weighted 1. Through a basic count we have the following values

$$c_{1,1} = 6 \quad c_{1,2} = 1 \quad c_{2,2} = 15 \quad c_{2,3} = 6 \quad c_{3,3} = 20 \quad c_{3,4} = 15 \quad c_{4,4} = 15 \quad c_{4,5} = 20$$

$$c_{5,5} = 6 \quad c_{5,6} = 15 \quad c_{6,6} = 1 \quad c_{6,7} = 1$$

Thus, the weighted vertex clique polynomial of $H$ is

$$p_H(t) = 1 - 6t - t^2 + 15t^2 + 6t^3 - 20t^3 - 15t^4 + 15t^4 + 20t^5 - 6t^5 - 15t^6 + t^6 + 6t^7 - t^8$$

$$= 1 - 6t + 14t^2 - 14t^3 + 14t^5 - 14t^6 + 6t^7 - t^8.$$  

We would like to remove the terms with degree larger than 3. Adding 20 vertices, 14 with weight 5 and 6 vertices with weight 7 removes the $t^5$ term and the $t^7$ term from the above polynomial. Adding 14 edges between our weight 5 vertices with a single weight 1 vertex removes the $t^6$ term without introducing additional complete graphs larger than edges. Adding a single edge between a weight 1 vertex and a weight 7 vertex removes the $t^8$ term. Thus,

$$c_\Gamma(t) = 1 - 6t + 14t^2 - 14t^3.$$  

So, by Theorem 3.2.20, the Maclaurin series expansion of

$$\frac{1}{H_R(-t)} = \frac{1}{1 - 6t + 14t^2 - 14t^3}.$$
has all positive coefficients.

We now show \( \epsilon_3' = 0 \). Expanding \( \frac{1}{H_R(-t)} \) yields

\[
\frac{1}{H_R(-t)} = 1 + 6t + 22t^2 + 62t^3 + \cdots.
\]

By Remark 3.2.11,

\[
\epsilon_1' = 6 \quad \epsilon_2' = 22 - \binom{6}{2} = 7 \quad \epsilon_3' = 62 - (7)(6) - \binom{6}{3} = 0.
\]

Hence, \( R \) satisfies (c) (since \( R \) is not a complete intersection).

We now aim to calculate the the Betti table for \( R \) resolved over \( S \). The unreduced Hilbert series reveals the following partial Betti table

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>-</td>
<td>7</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>-</td>
<td>-</td>
<td>21</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>3</td>
<td>-</td>
<td>-</td>
<td>*</td>
<td>*</td>
<td>*</td>
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<td>*</td>
</tr>
</tbody>
</table>

Consider the short exact sequence

\[
0 \to S/((q_1,\ldots,q_6) : (q_7))(-2) \to S/(q_1,\ldots,q_6) \to S/J \to 0. \tag{3.2}
\]

Taking a long exact sequence of Tor yields

\[
\cdots \to \text{Tor}_i(S/(q_1,\ldots,q_6), C)_j \to \text{Tor}_i(S/J, C)_j \to \text{Tor}_{i-1}(S/((q_1,\ldots,q_6) : (q_7)), C)_{j-2} \to \text{Tor}_{i-1}(S/(q_1,\ldots,q_6), C)_j \to \cdots.
\]

Letting \( i = 3 \) and \( j = 5 \) yields

\[
\beta^S_{3,5}(S/J) = \beta^S_{2,3}(S/((q_1,\ldots,q_6) : (q_7))).
\]
Furthermore, since \((q_1, \ldots, q_7)\) is an almost complete intersection, then \(S/((q_1, \ldots, q_6) : (q_7))\) is Gorenstein (Huneke and Ulrich (1987)). So,

\[
\beta^{S}_{2,3}(S/(q_1, \ldots, q_6) : (q_7)) = \beta^{S}_{4,7}(S/(q_1, \ldots, q_6) : (q_7)).
\]

Using our exact sequence of Tor and letting \(i = 5\) and \(j = 9\), yields

\[
\beta^{S}_{4,7}(S/(q_1, \ldots, q_6) : (q_7)) = \beta^{S}_{2,3}(S/(q_1, \ldots, q_6) : (q_7)) = \beta^{S}_{3,5}(S/J) = 0.
\]

So, the Betti table for \(S/J\) is

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
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<tr>
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<td>-</td>
</tr>
<tr>
<td>2</td>
<td>-</td>
<td>-</td>
<td>21</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>-</td>
<td>-</td>
<td>14</td>
<td>105</td>
<td>132</td>
<td>70</td>
<td>14</td>
</tr>
</tbody>
</table>

In particular, \(\beta^{S}_{2,5}(S/J) \neq 0\). Thus, \((d)\) is satisfied. Moreover, since

\[
21 = \binom{7}{2} = \beta^{S}_{2,4}(S/J),
\]

and \(J\) is not a complete intersection, \((b)\) is satisfied.
Below is a diagram summarizing the lack of implications, and one implication.

A dashed line is indicating an example showing that one condition does not imply the other. Furthermore, we leave the implication (b) implies (c) open.

### 3.3 Koszul Filtrations

Because of the difficulty in demonstrating rings are Koszul, we sometimes turn to a divide-and-conquer strategy to prove algebras are Koszul.

**Definition 3.3.1.** (Conca et al. (2001b)) A **Koszul filtration** $\mathcal{F}$ of a $\mathbb{C}$-algebra $R$ is a set of ideals of $R$ such that:

(a) Every ideal $I \in \mathcal{F}$ is generated by elements of degree 1.

(b) The zero ideal $(0)$ and the graded maximal ideal $m_R$ are in $\mathcal{F}$.

(c) For every $I \in \mathcal{F}$ with $I \neq 0$, there exists $J \in \mathcal{F}$ such that $J \subseteq I$, $I/J$ is cyclic, and $J : I \in \mathcal{F}$.

This notion was inspired by the work of Herzog, Hibi, and Restuccia on strongly Koszul algebras (Herzog et al. (2000)). Conca, Trung, and Valla, showed that the existence of a Koszul filtration in
$R$ implies that every ideal in the filtration has a linear resolution over $R$; in particular $R$ is Koszul, since $\mathbb{C} \cong R/m_R$.

**Theorem 3.3.2.** (Conca et al. (2001b)) Let $\mathcal{F}$ be a Koszul filtration of a standard graded $\mathbb{C}$-algebra $R$. Then the following hold:

(a) For every $J \in \mathcal{F}$ the $R$-module, $R/J$ is Koszul.

(b) In particular, $R$ is Koszul.

**Proof.** Suppose that $\mathcal{F}$ is a Koszul filtration of $R$ and $I, J \in \mathcal{F}$ such that $I/J = (\ell)$. Consider the short exact sequence

$$0 \to R/(J : \ell)(-1) \to R/J \to R/I \to 0.$$ 

Taking a long exact sequence of Tor yields the following

$$\cdots \to \text{Tor}_i^R(R/J, \mathbb{C})_j \to \text{Tor}_i^R(R/I, \mathbb{C})_j \to \text{Tor}_{i-1}^R(R/(J : \ell), \mathbb{C})_{j-1} \to \cdots,$$  \hspace{1cm} (3.3)

for every $j \in \mathbb{Z}$.

By induction on the number of generators of $J$ and the homological degree, we have the following

$$\text{Tor}_i^R(R/J, \mathbb{C})_j = 0 \quad \text{and} \quad \text{Tor}_{i-1}^R(R/(J : \ell), \mathbb{C})_{j-1} = 0,$$

for $i \neq j$. Thus, the first and third terms of Equation 3.3 vanish for all $i \neq j$. The second claim immediately follows from the first, since $m_R \in \mathcal{F}$ and $R/m_R \cong \mathbb{C}$. \hfill $\Box$

An interesting question to consider is: if a $\mathbb{C}$-algebra is Koszul, then does it admit a Koszul filtration? Unfortunately, the answer to this question is no.

**Example 3.3.3.** (Conca et al. (2001a)) Suppose $S = \mathbb{C}[x_0, \ldots, x_4]$ and

$$J = (x_0^2 - x_2x_3, x_1^2 - x_0x_4, x_2^2 - x_1x_4, x_3^2 - x_0x_1, x_4^2 - 3x_0x_3).$$

The algebra $R$ has the Hilbert series

$$H_R(t) = 1 + 5t + 10t^2 + 10t^3 + 5t^4 + t^5.$$
Thus, $R$ is Artinian. Furthermore, $R = S/J$ is Koszul, since it is a complete intersection. Now, a necessary condition for an Artinian algebra to have a filtration is that the defining ideal contains a quadric of rank 1 or 2. Indeed, every nonzero linear combination of the above quadratic forms is a quadric of rank at least three.

We can now finally provide a very elegant proof of 3.1.5.

**Theorem 3.1.5.** (Fröberg (1999)) Let $R = S/J$ and $J$ an ideal generated by monomials of degree 2. Then $R$ is Koszul.

**Proof.** Define $F$ to be the set of all ideals in $R$ generated by indeterminates and suppose that $J = (f_1, \ldots, f_m)$, where each $f_i$ is a quadratic monomial. For any ideal $P \subseteq R$ generated by indeterminates and $x_j \in P$, the following equality holds

$$P : (x_j) = P + (x_i \mid x_i \text{ divides some } f_k \text{ in } S).$$

Thus, $R$ is Koszul by Theorem 3.3.2. \hfill \Box

### 3.4 G-Quadartic and LG-quadratic Algebras

Another strategy to prove an algebra is Koszul is to prove something stronger.

**Definition 3.4.1.** We say $R = S/J$ is **G-quadratic** if the defining ideal $J$ has a Gröbner basis of quadrics with respect to some coordinate system of $S_1$ and some term order $\tau$ on $S$. An algebra $R$ is **LG-quadratic** if there exists a G-quadratic algebra $A$ and a regular sequence of linear forms $\ell_1, \ldots, \ell_c$ such that $R \cong A/(\ell_1, \ldots, \ell_c)$.

**Remark 3.4.2.** It is immediate that every G-quadratic algebra is LG-quadratic.

At first, similar to the Koszul property, it may seem that the G-quadratic property is rather restrictive, but it turns out that many classical algebras in their standard coordinate systems are G-quadratic, including coordinate rings of Grassmannians, Schubert varieties, Flag varieties, Hibi rings, and many more. The disadvantage of the G-quadratic property is that it can be rather
difficult to show algebras are G-quadratic; nevertheless, showing an algebra is G-quadratic can be a useful tool for demonstrating an algebra is Koszul. Before we prove every G-quadratic algebra is Koszul we need a theorem and a definition.

**Definition 3.4.3.** Given a fixed monomial order $<$ on $S$, the initial monomial of a nonzero polynomial $f \in S$ is the largest monomial $\text{in}_<(f)$ appearing in $f$ with a nonzero coefficient. The initial ideal of an ideal $I \subseteq S$ is the monomial ideal

$$\text{in}_<(I) = (\text{in}_<(f) : f \in I).$$

**Theorem 3.4.4.** *(Peeva (2011))* Suppose $I, I_1,$ and $I_2$ are graded ideals in $S$ such that $I \subseteq I_1$ and $I \subseteq I_2$. If $<$ is any monomial order on $S$, then for all $i, j$ we have

$$\dim \text{Tor}^S_i(S/I_1, S/I_2)_j \leq \dim \text{Tor}^S_i(S/\text{in}_<(I_1), S/\text{in}_<(I_2))_j.$$ 

**Theorem 3.4.5.** If $R$ is a G-quadratic algebra, then $R$ is Koszul

*Proof.* Let $I_1 = I_2 = m_S$. Applying Theorem 3.4.4 yields the following

$$\beta^{R(C)}_{i,j} \leq \beta^{S/\text{in}_<(I)}_{i,j}(C),$$

for all $i$ and $j$. By Theorem 3.1.5, $\beta^{S/\text{in}_<(I)}_{i,j}(C) = 0$, for $i \neq j$, since $\text{in}_<(I)$ is a quadratic monomial ideal. \qed

There is an interesting relationship between an algebra and a quotient of that algebra.

**Proposition 3.4.6.** *(Conca et al. (2013); Peeva (2011))* Let $A$ be a standard graded $\mathbb{C}$-algebra, $B$ a quotient of $A$, and $\ell$ is a regular element of degree 1 or 2.

(a) If $A$ is Koszul and $\text{reg}_A(B) \leq 1$, then $B$ is Koszul.

(b) If $B$ is Koszul and $\text{reg}_A(B)$ is finite, then $A$ is Koszul.

(c) Either $A$ and $A/\ell$ are both Koszul or both not Koszul.

**Theorem 3.4.7.** *(Conca et al. (2013))* If $R$ is an LG-quadratic algebra, then $R$ is Koszul.
**Proof.** Suppose that $R$ is LG-quadratic. By definition,

$$R \cong A/(\ell_1, \ldots, \ell_c),$$

where $\ell_1, \ldots, \ell_c$ is a regular sequence of linear forms. By Theorem 3.4.5, the $\mathbb{C}$-algebra $A$ is Koszul. Thus, induction and Proposition 3.4.6 yield the result.

An interesting class of algebras which are LG-quadratic is a complete intersection of quadrics.

**Proposition 3.4.8 (Caviglia (2004)).** Let $R = S/J$ be a complete intersection of quadrics. Then $R$ is LG-quadratic.

**Proof.** Suppose that $R = \mathbb{C}[x_0, \ldots, x_n]/(q_1, \ldots, q_m)$ is a complete intersection of quadrics. Set

$$A = \mathbb{C}[y_1, \ldots, y_n]/(y_1^2 + q_1, \ldots, y_m^2 + q_m),$$

and notice that $A$ is G-quadratic, because the initial ideal of its defining ideal with respect to the lex term order $y_i > x_j$ for every $i$ and $j$ is $(y_1^2, \ldots, y_m^2)$. Furthermore, $y_1, \ldots, y_m$ is an $A$-sequence, which proves the claim.

LG-quadratic algebras play a very important role in the study of Koszul algebras because of the strong restrictions they have concerning their Betti numbers over $S$. Indeed, suppose $R = S/J$ is a G-quadratic algebra and $Q$ is a quadratic initial ideal of $J$. Since, $J$ is generated by quadrics and

$$H_R(t) = H_{S/Q}(t),$$

then $\beta_1(R) = \beta_1(S/Q)$. Consequently, we have the following inequality

$$\beta_i(R) \leq \beta_i(S/Q) \leq \binom{g}{i}$$

for every $i \geq 0$. Since quotients by a regular sequence of linear forms do not affect the Betti numbers of $R$ over $S$, we get the following remarkable theorem.
Theorem 3.4.9. If $R = S/J$ is LG-quadratic and $J$ is minimally generated by $g$ elements, then the following inequality holds

$$\beta_i(R) \leq \binom{g}{i},$$

for $i \leq \pdim_S(R)$.

Remark 3.4.10. It is an open question if the previous theorem holds for Koszul algebras.

In general, we have the following implications of the previous properties.

$$\text{G-quadratic} \Rightarrow \text{LG-quadratic} \Rightarrow \text{Koszul} \Rightarrow \text{Quadratic Algebras}$$

Each of these implications is strict. In fact, we have already seen some examples demonstrating these implications are not reversible. We present two examples showing the remaining two implications are not reversible and note that Example 3.1.4 demonstrates that not all quadratic algebras are Koszul.

Example 3.4.11 (Conca et al. (2013)). Let

$$R = \mathbb{C}[x_0, x_1, x_2, x_3]/(x_0 x_1, x_0 x_3, (x_0 - x_1)x_2, x_2^2, x_0^2 + x_2 x_3).$$

The Hilbert series of $R$ is

$$\frac{1 + 2t - 2t^2 - 2t^3 + 2t^4}{(1-t)^2}.$$  

The algebra $R$ is Koszul since it has the following Koszul filtration:

$$\mathcal{F} = \{(0_R), (x_0), (x_1), (x_1, x_3), (x_0, x_1), (x_0, x_2), (x_0, x_1, x_3), (x_0, x_1, x_2), (x_0, x_1, x_2, x_3)\},$$

and $\mathcal{F}$ has the following computed colon ideals

$$(x_0, x_1, x_2) : (x_0, x_1, x_2, x_3) = (x_0, x_1, x_2)$$

$$(x_0, x_1, x_3) : (x_0, x_1, x_2, x_3) = (x_0, x_1, x_2, x_3)$$

$$(x_0, x_1) : (x_0, x_1, x_3) = (x_0, x_1, x_2)$$

$$(x_1, x_3) : (x_0, x_1, x_3) = (x_0, x_1, x_2, x_3)$$

$$(x_0, x_2) : (x_0, x_1, x_2) = (x_0, x_2)$$
(x_1) : (x_1, x_3) = (x_0, x_1) \\
(x_0) : (x_0, x_1) = (x_0, x_2) \\
(0_R) : (x_0) = (x_1, x_3) \\
(0_R) : (x_1) = (x_0).

This algebra is not LG-quadratic. First, recall that the \(h\)-polynomial does not change under lifting by a regular sequence of linear forms. Hence, to prove that \(R\) is not LG-quadratic it is enough to verify that there is no algebra with quadratic monomial relations with \(h\)-polynomial

\[
h(t) = 1 + 2t - 2t^2 - 2t^3 + 2t^4.
\]

In general, if \(I\) is an ideal of \(S\), not containing linear forms, with \(h\)-polynomial

\[
h(t) = 1 + h_1 t + h_2 z^2 + \cdots,
\]

then \(I\) has codimension \(h_1\) and exactly \((h_1 + 1) - h_2\) quadratic generators. So, we seek an ideal with codimension 2 and 5 monomial generators chosen among the generators \((x_0, x_1)(x_0, \ldots, x_6)\). An exhaustive search in Macaulay2 (Grayson and Stillman) verifies that no such ideal exists.

**Remark 3.4.12.** This is the only known commutative example of a Koszul algebra which is not LG-quadratic. There is a non-commutative example over an exterior algebra (McCullough and Mere (2022)). Thus, it would be very interesting to produce a family of commutative non-obstructed Koszul algebras which are not LG-quadratic, similar to Roos’s family.

**Example 3.4.13 (Conca (2014)).** We now present a \(\mathbb{C}\)-algebra that admits the LG-quadratic property but not the G-quadratic property. Let

\[
R = \mathbb{C}[x_0, x_1, x_2, x_3] / (x_0^2 - x_1 x_2, x_2 x_3, x_1^2, x_0 x_2, x_0 x_1).
\]

and

\[
R' = \mathbb{C}[x_0, x_1, x_2, x_3, x_4] / (x_0^2 - x_1 x_2 + x_1 x_4, x_3^2, x_2 x_3, x_1^2 + x_1 x_4, x_0 x_2, x_0 x_1 + x_0 x_4).
\]
The Hilbert series of $R$ is
\[
\frac{1 + 3t - 3t^2}{1 - t}.
\]
An exhaustive search in Macaulay2 verifies that $h(t) = 1 + 3t - 3t^2$ is not the $h$-polynomial of any quadratic monomial ideal in four variables. Thus, $R$ cannot be G-quadratic. However, $R'$ is G-quadratic since the initial ideal of the defining ideal is $(x_0^2, x_1^2, x_0x_3, x_2x_3, x_1x_4, x_3x_4)$, under the rev-lex order. Furthermore, we have the following isomorphism
\[
R \cong \mathbb{C}[x_0, x_1, x_2, x_3, x_4]/(x_0^2 - x_1x_2 + x_1x_4, x_3, x_2x_3, x_1^2 + x_1x_4, x_0x_2, x_0x_1 + x_0x_4) + (x_4),
\]
where $x_4$ is a regular on $R'$. So $R$ is LG-quadratic, but not G-quadratic.

### 3.5 References


CHAPTER 4. GENERIC LINES IN PROJECTIVE SPACE AND THE KOSZUL PROPERTY

Modified from a manuscript published in Nagoya Mathematical Journal

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4.1 Abstract

In this paper, we study the Koszul property of the homogeneous coordinate ring of a generic collection of lines in $\mathbb{P}^n$ and the homogeneous coordinate ring of a collection of lines in general linear position in $\mathbb{P}^n$. We show that if $\mathcal{M}$ is a collection of $m$ lines in general linear position in $\mathbb{P}^n$ with $2m \leq n + 1$ and $R$ is the coordinate ring of $\mathcal{M}$, then $R$ is Koszul. Further, if $\mathcal{M}$ is a generic collection of $m$ lines in $\mathbb{P}^n$ and $R$ is the coordinate ring of $\mathcal{M}$ with $m$ even and $m + 1 \leq n$ or $m$ is odd and $m + 2 \leq n$, then $R$ is Koszul. Lastly, we show if $\mathcal{M}$ is a generic collection of $m$ lines such that

$$m > \frac{1}{12} \left(3(n^2 + 10n + 13) + \sqrt{3(n-1)^3(3n+5)}\right),$$

then $R$ is not Koszul. We give a complete characterization of the Koszul property of the coordinate ring of a generic collection of lines for $n \leq 6$ or $m \leq 6$. We also determine the Castelnuovo-Mumford regularity of the coordinate ring for a generic collection of lines and the projective dimension of the coordinate ring of collection of lines in general linear position.

Keywords: generic lines, Koszul algebras, free resolutions, Castelnuovo-Mumford regularity

4.2 Introduction

Let $S = \mathbb{C}[x_0, \ldots, x_n]$ be a polynomial ring and $J$ a graded homogeneous ideal of $S$. Following Priddy’s work, we say the ring $R = S/J$ is Koszul if the minimal graded free resolution of the
field $\mathbb{C}$ over $R$ is linear (Priddy (1970)). Koszul rings are ubiquitous in commutative algebra. For example, any polynomial ring, all quotients by quadratic monomial ideals, all quadratic complete intersections, the coordinate rings of Grassmannians in their Plücker embedding, and all suitably high Veronese subrings of any standard graded algebra are all Koszul (Backelin and Fröberg (1985)). Because of the ubiquity of Koszul rings, it is of interest to determine when we can guarantee a coordinate ring will be Koszul. In 1992, Kempf proved the following theorem

**Theorem 4.2.1.** (Kempf (1992)) Let $\mathcal{P}$ be a collection of $p$ points in $\mathbb{P}^n$ and $R$ the coordinate ring of $\mathcal{P}$. If the points of $\mathcal{P}$ are in general linear position and $p \leq 2n$, then $R$ is Koszul.

In 2001, Conca, Trung, and Valla proved a similar theorem, except for a generic collection of points.

**Theorem 4.2.2.** (Conca et al. (2001)) Let $\mathcal{P}$ be a generic collection of $p$ points in $\mathbb{P}^n$ and $R$ the coordinate ring of $\mathcal{P}$. Then $R$ is Koszul if and only if $p \leq 1 + n + \frac{n^2}{4}$.

We aim to generalize these theorems to collections of lines. In Section 4.3, we review necessary background information and results related to Koszul algebras that we use in the other sections. In Section 4.4, we study properties of coordinate rings of collections of lines and how they differ from coordinate rings of collections of points. In particular, we show

**Theorem 4.4.5.** Let $\mathcal{M}$ be a generic collection of $m$ lines in $\mathbb{P}^n$ with $n \geq 3$ and $R$ the coordinate ring of $\mathcal{M}$. Then $\operatorname{reg}_S(R) = \alpha$, where $\alpha$ is the smallest non-negative integer such that \( \binom{n+\alpha}{\alpha} \geq m(\alpha + 1) \).

In Section 4.5, we prove

**Theorem 4.5.3.** Let $\mathcal{M}$ be a generic collection of $m$ lines in $\mathbb{P}^n$ such that $m \geq 2$ and $R$ the coordinate ring of $\mathcal{M}$.

(a) If $m$ is even and $m + 1 \leq n$, then $R$ has a Koszul filtration.

(b) If $m$ is odd and $m + 2 \leq n$, then $R$ has a Koszul filtration.
In particular, \( R \) is Koszul in these cases.

Additionally, we show the coordinate ring of a generic collection of 5 lines in \( \mathbb{P}^6 \) is Koszul by constructing a Koszul filtration. In Section 4.6, we prove

**Theorem 4.6.2.** Let \( \mathcal{M} \) be a generic collection of \( m \) lines in \( \mathbb{P}^n \) and \( R \) the coordinate ring of \( \mathcal{M} \). If

\[
m > \frac{1}{72} \left( 3(n^2 + 10n + 13) + \sqrt{3(n - 1)^3(3n + 5)} \right),
\]

then \( R \) is not Koszul.

Further, there is an exceptional example of a coordinate ring that is not Koszul; if \( \mathcal{M} \) is a collection of 3 lines in general linear position in \( \mathbb{P}^4 \), then the coordinate ring \( R \) is not Koszul. In Section 4.7, we exhibit a collection of lines that is not a generic collection but the lines are in general linear position, and we give two examples of coordinate rings where each define a generic collection of lines with quadratic defining ideals but for numerical reasons each coordinate ring is not Koszul.

We end the document with a table summarizing the results of which coordinate rings are Koszul, which are not Koszul, and which are unknown.

### 4.3 Background

Let \( \mathbb{P}^n \) denote \( n \)-dimensional projective space obtained from a \( \mathbb{C} \)-vector space of dimension \( n+1 \). A commutative Noetherian \( \mathbb{C} \)-algebra \( R \) is said to be **graded** if \( R = \bigoplus_{i \in \mathbb{N}} R_i \) as an Abelian group such that for all non-negative integers \( i \) and \( j \) we have \( R_i R_j \subseteq R_{i+j} \), and is **standard graded** if \( R_0 = \mathbb{C} \) and \( R \) is generated as a \( \mathbb{C} \)-algebra by a finite set of degree 1 elements. Additionally, an \( R \)-module \( M \) is called graded if \( R \) is graded and \( M \) can be written as \( M = \bigoplus_{i \in \mathbb{N}} M_i \) as an Abelian group such that for all non-negative integers \( i \) and \( j \) we have \( R_i M_j \subseteq M_{i+j} \). Note each summand \( R_i \) and \( M_i \) is a \( \mathbb{C} \)-vector space of finite dimension. We always assume our rings are standard graded.

Let \( S \) be the symmetric algebra of \( R_1 \) over \( \mathbb{C} \); i.e. \( S \) is the polynomial ring \( S = \mathbb{C}[x_0, \ldots, x_n] \), where \( \dim(R_1) = n + 1 \) and \( x_0, \ldots, x_n \) is a \( \mathbb{C} \)-basis of \( R_1 \). We have an induced surjection \( S \to R \) of standard graded \( \mathbb{C} \)-algebras, and so \( R \cong S/J \), where \( J \) is a homogenous ideal and the kernel of
This map. We say that $J$ defines $R$ and call this ideal $J$ the defining ideal. Denote by $m_R$ the maximal homogeneous ideal of $R$. Except when explicitly said, all rings are graded and Noetherian and all modules are finitely generated. We may view $C$ as a graded $R$-module since $C \cong R/m_R$.

The function $\text{Hilb}_M : \mathbb{N} \to \mathbb{N}$ defined by $\text{Hilb}_M(d) = \dim(C_d)$ is called the Hilbert function of the $R$-module $M$. Further, there exists a unique polynomial $\text{HilbP}(d)$ with rational coefficients, called the Hilbert polynomial such that $\text{HilbP}(d) = \text{Hilb}(d)$ for $d \gg 0$.

The minimal graded free resolution $F$ of an $R$-module $M$ is an exact sequence of homomorphisms of finitely generated free $R$-modules

$$F : \cdots \to F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \to F_1 \xrightarrow{d_1} F_0,$$

such that $d_{i-1}d_i = 0$ for all $i$, $M \cong F_0/\text{Im}(d_1)$, and $d_{i+1}(F_{i+1}) \subseteq (x_0, \ldots, x_n)F_i$ for all $i \geq 0$. After choosing bases, we may represent each map in the resolution as a matrix. We can write $F_i = \bigoplus_j R(-j)^{\beta_{i,j}^R(M)}$, where $R(-j)$ denotes a rank one free module with a generator in degree $j$, and the numbers $\beta_{i,j}^R(M)$ are called the graded Betti numbers of $M$ and are numerical invariants of $M$. The total Betti numbers of $M$ are defined as $\beta_i^R(M) = \sum_j \beta_{i,j}^R(M)$. When it is clear which module we are speaking about, we will write $\beta_{i,j}$ and $\beta_i$ to denote the graded Betti numbers and total Betti numbers, respectively. By construction, we have the equalities

$$\beta_i^R(M) = \dim \text{Tor}_i^R(M, C),$$
$$\beta_{i,j}^R(M) = \dim \text{Tor}_i^R(M, C)_j.$$

Two more invariants of a module are its projective dimension and relative Castelnuovo-Mumford regularity. These invariants are defined for an $R$-module $M$ as follows:

$$\text{pdim}_R(M) = \sup\{i \mid F_i \neq 0\} = \sup\{i \mid \beta_i(M) \neq 0\},$$
$$\text{reg}_R(M) = \sup\{j - i \mid \beta_{i,j}(M) \neq 0\}.$$
Certain invariants are related to one another. For example, if \( \text{pdim}_R(M) \) is finite, then the Auslander-Buchsbaum formula relates the projective dimension to the depth of a module Peeva (2011), where the depth of an \( R \)-module \( M \) is the length of the largest \( M \)-regular sequence consisting of elements of \( R \), and is denoted \( \text{depth}(M) \). Letting \( R = S \), the Auslander-Buchsbaum formula states that the projective dimension and depth of an \( S \)-module \( M \) are complementary to one another:

\[
\text{pdim}_S(M) + \text{depth}(M) = n + 1.
\] (4.1)

The Krull dimension, or dimension, of a ring is the supremum of the lengths \( k \) of strictly increasing chains \( P_0 \subset P_1 \subset \ldots \subset P_k \) of prime ideals of \( R \). The dimension of an \( R \)-module is denoted \( \text{dim}(M) \) and is the Krull dimension of the ring \( R/I \), where \( I = \text{Ann}_R(M) \) is the annihilator of \( M \). The depth and dimension of a ring have the following properties along a short exact sequence.

**Proposition 4.3.1.** (Eisenbud (1995)) Let \( R \) be a graded Noetherian ring and suppose that

\[
0 \to M' \to M \to M'' \to 0
\]

is an exact sequence of finitely generated graded \( R \)-modules. Then

(a) \( \text{depth}(M') \geq \min\{\text{depth}(M), \text{depth}(M'') + 1\} \),

(b) \( \text{depth}(M) \geq \min\{\text{depth}(M'), \text{depth}(M'')\} \),

(c) \( \text{depth}(M'') \geq \min\{\text{depth}(M), \text{depth}(M') - 1\} \),

(d) \( \text{dim}(M) = \max\{\text{dim}(M''), \text{dim}(M')\} \).

Furthermore, \( \text{depth}(M) \leq \text{dim}(M) \).

An \( R \)-module \( M \) is Cohen-Macaulay, if \( \text{depth}(M) = \text{dim}(M) \). Since \( R \) is a module over itself, we say \( R \) is a Cohen-Macaulay ring if it is a Cohen-Macaulay \( R \)-module. Cohen-Macaulay rings have been studied extensively, and the definition is sufficiently general to allow a rich theory with a wealth of examples in algebraic geometry. This notion is a workhorse in commutative algebra, and
provides very useful tools and reductions to study rings (Bruns and Herzog (1993)). For example, if one has a graded Cohen-Macaulay C-algebra, then one can take a quotient by generic linear forms to produce an Artinian ring. A reduction of this kind is called an Artinian reduction and provides many useful tools to work with, and almost all homological invariants of the ring are preserved (Migliore and Patnott (2011)). Unfortunately, we will not be able to use these tools or reductions as the coordinate ring of a generic collection of lines is almost never Cohen-Macaulay, whereas the coordinate ring of a generic collection of points is always Cohen-Macaulay.

The absolute Castelnuovo-Mumford regularity, or the regularity, is denoted \( \text{reg}_S(M) \) and is the regularity of \( M \) as an \( S \)-module. There is a cohomological interpretation by local duality (Eisenbud and Goto (1984)). Set \( H^i_{m_S}(M) \) to be the \( i \)th local cohomology module with support in the graded maximal ideal of \( S \). One has \( H^i_{m_S}(M) = 0 \) if \( i < \text{depth}(M) \) or \( i > \text{dim}(M) \) and

\[
\text{reg}_S(M) = \max\{j + i : H^i_{m_S}(M)_{j} \neq 0\}.
\]

In practice, bounding the regularity of \( M \) is difficult, since it measures the largest degree of a minimal syzygy of \( M \). We have tools to help the study of the regularity of an \( S \)-module.

**Proposition 4.3.2.** (Eisenbud (1995)) Suppose that

\[
0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0
\]

is an exact sequence of finitely generated graded \( S \)-modules. Then

(a) \( \text{reg}_S(M') \leq \max\{\text{reg}_S(M), \text{reg}_S(M'') + 1\} \)

(b) \( \text{reg}_S(M) \leq \max\{\text{reg}_S(M'), \text{reg}_S(M'')\} \)

(c) \( \text{reg}_S(M'') \leq \max\{\text{reg}_S(M), \text{reg}_S(M') - 1\} \)

and if \( d_0 = \min\{d \mid \text{Hilb}(d) = \text{HilbP}(d)\} \), then \( \text{reg}(M) \geq d_0 \). Furthermore, if \( M \) is Cohen-Macaulay, then \( \text{reg}_S(M) = d_0 \). If \( M \) has finite length, then \( \text{reg}_S(M) = \max\{d : M_d \neq 0\} \).
To study these invariants, we place the graded Betti numbers of a module $M$ into a table, called the Betti table

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\beta_{0,0}$</td>
<td>$\beta_{1,1}$</td>
<td>$\beta_{2,2}$</td>
<td>$\beta_{3,3}$</td>
<td>$\beta_{4,4}$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>1</td>
<td>$\beta_{0,1}$</td>
<td>$\beta_{1,2}$</td>
<td>$\beta_{2,3}$</td>
<td>$\beta_{3,4}$</td>
<td>$\beta_{4,5}$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td></td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\ddots$</td>
</tr>
</tbody>
</table>

The Betti table allows us to determine certain invariants easier; e.g., the projective dimension is the length of the table and the regularity is the height of the table.

Denote by $H_M(t)$ and $P^R_M(t)$ respectively the Hilbert series of $M$ and the Poincaré series of an $R$-module $M$:

$$H_M(t) = \sum_{i \geq 0} \text{Hilb}_M(i)t^i$$

and

$$P^R_M(t) = \sum_{i \geq 0} \beta^R_i(M)t^i.$$  

It is worth observing that since $M$ is finitely generated by homogenous elements of positive degree, the Hilbert series of $M$ is a rational function. A short exact sequence of modules has a property we use extensively in this paper. If we have a short exact sequence of graded $S$-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

then

$$H_B(t) = H_A(t) + H_C(t).$$

Whenever we use this property, we will refer to it as the additivity property of the Hilbert series.

A standard graded $C$-algebra $R$ is Koszul if $C$ has a linear $R$-free resolution; that is, $\beta^R_{i,j}(C) = 0$ for $i \neq j$. Koszul algebras possess remarkable homological properties. For example

**Theorem 4.3.3.** *(Avramov and Eisenbud (1992); Avramov and Peeva (2001))* The following are equivalent:
(a) Every finitely generated $R$-module has finite regularity.

(b) The residue field has finite regularity.

(c) $R$ is Koszul.

Koszul rings possess other interesting properties as well. Fröberg showed in Fröberg (1999) that $R$ is Koszul if and only if $H_R(t)$ and the $P^R_C(t)$ have the following relationship

$$P^R_C(t)H_R(-t) = 1.$$  \hfill (4.2)

In general, the Poincaré series of $C$ as an $R$-module can be irrational Anick (1982), but if $R$ is Koszul, then Equation (4.2) tells us the Poincaré series is always rational. So a necessary condition for a coordinate ring $R$ to be Koszul is $P^R_C(t) = \frac{1}{H_R(-t)}$ must have non-negative coefficients in its Maclaurin series. Another necessary condition is that if $R$ is Koszul, then the defining ideal has a minimal generating set of forms of degree at most 2. This is easy to see since

$$\beta^R_{2,j}(C) = \begin{cases} 
\beta^S_{1,j}(R) & \text{if } j \neq 2 \\
\beta^S_{1,2}(R) + \binom{n+1}{2} & \text{if } j = 2,
\end{cases}$$

Conca (2014). Unfortunately, the converse does not hold, but Fröberg showed that if the defining ideal is generated by monomials of degree at most 2, then $R$ is Koszul.

**Theorem 4.3.4.** (Backelin and Fröberg (1985)) If $R = S/J$ and $J$ is a monomial ideal with each monomial having degree at most 2, then $R$ is Koszul.

More generally, if $J$ has a Gröbner basis of quadrics in some term order, then $R$ is Koszul. If such a basis exists, we say that $R$ is **G-quadratic**. More generally, $R$ is **LG-quadratic** if there is a G-quadratic ring $A$ and a regular sequence of linear forms $l_1, \ldots, l_r$ such that $R \cong A/(l_1, \ldots, l_r)$. It is worth noting that every G-quadratic ring is LG-quadratic, and every LG-quadratic ring is Koszul and that all of these implications are strict (Conca (2014)). We briefly discuss in Section 4.7 if coordinate rings of generic collections of lines are G-quadratic or LG-quadratic.

We now define a very useful tool in proving rings are Koszul.
**Definition 4.3.5.** Let $R$ be a standard graded $\mathbb{C}$-algebra. A family $\mathcal{F}$ of ideals is said to be a **Koszul filtration** of $R$ if

(a) Every ideal $I \in \mathcal{F}$ is generated by linear forms,

(b) The ideal 0 and the maximal homogeneous ideal $m_R$ of $R$ belong to $\mathcal{F}$,

(c) For every ideal $I \in \mathcal{F}$ different from 0, there exists an ideal $K \in \mathcal{F}$ such that $K \subset I, I/K$ is cyclic, and $K : I \in \mathcal{F}$.

Conca, Trung, and Valla showed in Conca et al. (2001) that if $R$ has a Koszul filtration, then $R$ is Koszul. In fact, a stronger statement is true.

**Proposition 4.3.6.** *(Conca et al. (2001))* Let $\mathcal{F}$ be a Koszul filtration of $R$. Then $\text{Tor}_i^R(R/J, \mathcal{C})_j = 0$ for all $i \neq j$ and for all $J \in \mathcal{F}$. In particular, $R$ is Koszul.

Conca, Trung, and Valla construct a Koszul filtration to show certain sets of points in **general linear position** are Koszul in Conca et al. (2001). Since we aim to generalize Theorems 4.2.1 and 4.2.2 to collections of lines, we must define what it means for a collection of lines to be generic and what it means for a collection of lines to be in general linear position.

**Definition 4.3.7.** Let $\mathcal{P}$ be a collection of $p$ points in $\mathbb{P}^n$ and $\mathcal{M}$ be a collection of $m$ lines in $\mathbb{P}^n$. The points of $\mathcal{P}$ are in **general linear position** if any $s$ points span a $\mathbb{P}^r$, where $r = \min\{s - 1, n\}$. Similarly, the lines of $\mathcal{M}$ are in **general linear position** if any $s$ lines span a $\mathbb{P}^r$, where $r = \min\{2s - 1, n\}$. A collection of points in $\mathbb{P}^n$ is a **generic collection** if every linear form in the defining ideal of each point has algebraically independent coefficients over $\mathbb{Q}$. Similarly, we say a collection of lines is a **generic collection** if every linear form in the defining ideal of each line has algebraically independent coefficients over $\mathbb{Q}$.

We can interpret this definition as saying generic collections are sufficiently random since the collection of them forms a dense subset of large parameter space. Furthermore, as one should suspect, a generic collection of lines is in general linear position, since collections of lines in general
linear position are characterized by the nonvanishing of certain determinants in the coefficients of
the defining linear forms. The converse is not true; see Example 4.7.1.

Remark 4.3.8. Suppose $\mathcal{P}$ is a collection of $p$ points in general linear position in $\mathbb{P}^n$ and $\mathcal{M}$ is a
collection of $m$ lines in general linear position in $\mathbb{P}^n$. The defining ideal for each point is minimally
generated by $n$ linear forms and the defining ideal for each line is minimally generated by $n - 1$
linear forms. We can see this because a point is an intersection of $n$ hyperplanes and a line is an
intersection of $n - 1$ hyperplanes. Also, if $K$ is the defining ideal for $\mathcal{P}$ and $J$ is the defining ideal
for $\mathcal{M}$, then $\dim_{\mathbb{C}}(K_1) = n + 1 - p$ and $\dim_{\mathbb{C}}(J_1) = n + 1 - 2m$, provided either quantity is non-zero.

4.4 Properties of Coordinate Rings of Lines

This section aims to establish properties for the coordinate rings of generic collections of lines
and collections of lines in general linear position and compare them to the coordinate rings of
generic collections of points and collections of points in general linear position. We will see that the
significant difference between the two coordinate rings is that the coordinate ring $R$ of a collection
of lines in at least general linear position is never Cohen-Macaulay, unless $R$ is the coordinate ring
of a single line, while the coordinate rings of points in general linear position are always Cohen-
Macaulay. The lack of the Cohen-Macaulay property presents difficulty since many techniques are
not available to us, such as Artinian reductions.

Proposition 4.4.1. Let $\mathcal{M}$ be a collection of lines in general linear position in $\mathbb{P}^n$ with $n \geq 3$, and
$R$ the coordinate ring of $\mathcal{M}$. If $|\mathcal{M}| = 1$, then $\pdim_{\mathbb{S}}(R) = n - 1$, $\depth(R) = 2$, and $\dim(R) = 2$; if
$|\mathcal{M}| \geq 2$, then $\pdim_{\mathbb{S}}(R) = n$, $\depth(R) = 1$, and $\dim(R) = 2$. In particular, $R$ is Cohen-Macaulay
if and only if $|\mathcal{M}| = 1$.

Proof. We prove the claim by induction on $|\mathcal{M}|$. Let $m = |\mathcal{M}|$ and let $J$ be the defining ideal
of $\mathcal{M}$. If $m = 1$, then by Remark 4.3.8 the ideal $J$ is minimally generated by $n - 1$ linear forms.
So, $R$ is isomorphic to a polynomial ring in two indeterminates. Now, suppose that $m \geq 2$, and
write $J = K \cap I$, where $K$ is the defining ideal for $m - 1$ lines and $I$ is the defining ideal for the
remaining single line. By induction, \( \text{depth}(S/K) \leq 2 \) and \( \text{dim}(S/K) = \text{dim}(S/I) = 2 \). Furthermore, \( S/(K + I) \) is Artinian, since the variety \( K \) defines intersects trivially with the variety \( I \) defines. Hence, \( \text{dim}(S/(I + K)) = 0 \). So, by Proposition 4.3.1 the \( \text{depth}(S/(I + K)) = 0 \).

Using the short exact sequence

\[
0 \longrightarrow S/J \longrightarrow S/K \oplus S/I \longrightarrow S/(K + I) \longrightarrow 0,
\]

and Proposition 4.3.1, we have two inequalities

\[
\min \{ \text{depth}(S/K \oplus S/I), \text{depth}(S/(I + K)) + 1 \} \leq \text{depth}(S/J),
\]

and

\[
\min \{ \text{depth}(S/K \oplus S/I), \text{depth}(S/J) - 1 \} \leq \text{depth}(S/(I + K)).
\]

Regardless if \( \text{depth}(S/K) \) is 1 or 2, our two inequalities yield \( \text{depth}(S/J) = 1 \). By the Auslander–Buchsbaum formula, we have \( \text{pdim}_S(S/J) = n \). Lastly, Proposition 4.3.1, yields \( \text{dim}(S/J) = 2 \).

**Remark 4.4.2.** We would like to note that when \( n = 2 \), \( R \) is a hypersurface and so \( \text{pdim}_S(R) = 1 \), \( \text{depth}(R) = 2 \), and \( \text{dim}(R) = 2 \). Thus, we restrict our attention to the case \( n \geq 3 \). Furthermore, an identical proof shows that if \( \mathcal{P} \) is a collection of points in general linear position in \( \mathbb{P}^n \) and \( R \) is the coordinate ring of \( \mathcal{P} \), then \( \text{pdim}_S(R) = n \), \( \text{depth}(R) = 1 \), and \( \text{dim}(R) = 1 \). Hence, \( R \) is Cohen-Macaulay.

In Conca et al. (2001), Conca, Trung, and Valla used the Hilbert function of points in \( \mathbb{P}^n \) in general linear position to prove the corresponding coordinate ring is Koszul, provided the number of points is at most \( 2n + 1 \). There is a generalization for the Hilbert function to a generic collection of points. We present both together as a single theorem for completeness, we do not use the Hilbert function for a generic collection of points.

**Theorem 4.4.3.** (Carlini et al. (2012); Conca et al. (2001)) Suppose that \( \mathcal{P} \) is a collection of \( p \) points in \( \mathbb{P}^n \). If \( \mathcal{P} \) is a generic collection, or \( \mathcal{P} \) is a collection in general linear position with
\( p \leq 2n + 1 \), then the Hilbert function of \( R \) is
\[
\text{Hilb}_R(d) = \min \left\{ \binom{n + d}{d}, p \right\}.
\]

In particular, if \( p \leq n + 1 \), then
\[
H_R(t) = \frac{(p - 1)t + 1}{1 - t}.
\]

Since we aim to generalize Theorems 4.2.1 and 4.2.2, we would like to know the Hilbert series of the coordinate ring of a generic collection of lines. The famous Hartshorne-Hirschowitz Theorem provides an answer.

**Theorem 4.4.4.** *(Hartshorne and Hirschowitz (1982))* Let \( \mathcal{M} \) be a generic collection of \( m \) lines in \( \mathbb{P}^n \) and \( R \) the coordinate ring of \( \mathcal{M} \). The Hilbert function of \( R \) is
\[
\text{Hilb}_R(d) = \min \left\{ \binom{n + d}{d}, m(d + 1) \right\}.
\]

This theorem is very difficult to prove. One could ask if any generalization holds for planes, and unfortunately, this is not known and is an open problem. Interestingly, this theorem allows us to determine the regularity for the coordinate ring \( R \) of a generic collection of lines.

**Theorem 4.4.5.** Let \( \mathcal{M} \) be a generic collection of \( m \) lines in \( \mathbb{P}^n \) with \( n \geq 3 \) and \( R \) the coordinate ring of \( \mathcal{M} \). Then \( \text{reg}_S(R) = \alpha \), where \( \alpha \) is the smallest non-negative integer satisfying \( \binom{n + \alpha}{\alpha} \geq m(\alpha + 1) \).

**Proof.** If \( m = 1 \), then by Remark 4.3.8 and a change of basis we can write the defining ideal as \( J = (x_0, \ldots, x_{n-2}) \). The coordinate ring \( R \) is minimally resolved by the Koszul complex on \( x_0, \ldots, x_{n-2} \). So, \( \text{reg}_S(R) = 0 \), and this satisfies the inequality. Suppose that \( m \geq 2 \) and let \( \alpha \) be the smallest non-negative integer satisfying \( \binom{n + \alpha}{\alpha} \geq m(\alpha + 1) \). By Theorem 4.4.4 and Proposition 4.3.2, \( \text{reg}_S(R) \geq \alpha \).

We show the reverse inequality by induction on \( m \). Let \( J \) be the defining ideal for the collection \( \mathcal{M} \). Note, removing a line from a generic collection of lines maintains the generic property for the new collection. Let \( K \) be the defining ideal for \( m - 1 \) lines and \( I \) the defining ideal for the remaining
line such that \( J = K \cap I \). By induction \( \text{reg}_S(S/K) = \beta \), and \( \beta \) is the smallest non-negative integer satisfying the inequality \( \binom{n+\beta}{\beta} \geq (m-1)(\beta+1) \).

Now, we claim that \( \text{reg}_S(S/K) = \beta \in \{\alpha, \alpha-1\} \). To prove this we need two inequalities: \( m-2 \geq \beta \) and \( \binom{n+\beta}{\beta+1} \geq n(m-1) \). We have the first inequality since

\[
\binom{n+m-2}{m-2} - (m-1)(m-2+1) = \frac{(n+m-2)!}{n!(m-2)!} - (m-1)^2 = \frac{(m+1)!}{3!(m-2)!} - (m-1)^2 = \frac{(m-3)(m-2)(m-1)}{3!} \geq 0.
\]

Thus, \( m-2 \geq \beta \). We have the second inequality, since by assumption

\[
\binom{n+\beta}{\beta} \geq (m-1)(\beta+1),
\]

and rearranging terms gives

\[
\binom{n+\beta}{\beta+1} \geq n(m-1).
\]

These inequalities together yield the following

\[
\binom{n+\beta+1}{\beta+1} = \binom{n+\beta}{\beta} + \binom{n+\beta}{\beta+1} \geq (m-1)(\beta+1) + n(m-1) = (m-1)(\beta+1) + m + (m-1)(n-1) - 1 \geq (m-1)(\beta+1) + m + (m-1)2 - 1 \geq (m-1)(\beta+1) + m + \beta + 1 = m(\beta + 2).
\]

Hence, \( \beta + 1 \geq \alpha \). Furthermore, the inequality

\[
\binom{n+\beta-1}{\beta-1} < (m-1)(\beta-1+1) \leq m\beta
\]

implies that \( \alpha \geq \beta \). So, \( \text{reg}_S(S/K) = \beta \) where \( \beta \in \{\alpha, \alpha-1\} \).
Consider the short exact sequence

\[ 0 \rightarrow S/J \rightarrow S/K \oplus S/I \rightarrow S/(K + I) \rightarrow 0. \]

If \( \beta = \alpha \), then Theorem 4.4.4 and the additive property of the Hilbert series yields the following

\[
H_{S/(K+I)}(t) = (H_{S/K}(t) + H_{S/I}(t)) - H_{S/J}(t)
\]

\[
= \left( \sum_{k=0}^{\alpha-1} \binom{n+k}{k} t^k + \sum_{k=0}^{\infty} (m-1)(k+1)t^k + \sum_{k=0}^{\infty} (k+1)t^k \right)
- \left( \sum_{k=0}^{\alpha-1} \binom{n+k}{k} t^k - \sum_{k=\alpha}^{\infty} m(k+1)t^k \right)
= \sum_{k=0}^{\alpha-1} (k+1)t^k.
\]

and similarly if \( \beta = \alpha - 1 \), then

\[
H_{S/(K+I)}(t) = (H_{S/K}(t) + H_{S/I}(t)) - H_{S/J}(t)
\]

\[
= \left( \sum_{k=0}^{\alpha-2} \binom{n+k}{k} t^k + \sum_{k=\alpha-1}^{\infty} (m-1)(k+1)t^k + \sum_{k=0}^{\infty} (k+1)t^k \right)
- \left( \sum_{k=0}^{\alpha-1} \binom{n+k}{k} t^k - \sum_{k=\alpha}^{\infty} m(k+1)t^k \right)
= \sum_{k=0}^{\alpha-2} (k+1)t^k + \left( m\alpha - \binom{n+\alpha-1}{\alpha-1} \right)t^{\alpha-1}.
\]

Note \( m\alpha - \binom{n+\alpha-1}{\alpha-1} \) is positive since \( \alpha \) is the smallest non-negative integer such that

\[
\binom{n+\alpha}{\alpha} \geq m(\alpha + 1).
\]

So, \( S/(K + I) \) is Artinian. By Proposition 4.3.2, \( \operatorname{reg}_S(S/(K + I)) = \alpha - 1 \). Since \( \operatorname{reg}_S(S/K) = \alpha \) or \( \operatorname{reg}_S(S/K) = \alpha - 1 \) and \( \operatorname{reg}_S(S/I) = 0 \), then \( \operatorname{reg}_S(S/J) \leq \alpha \). Thus, \( \operatorname{reg}_S(R) = \alpha \).

**Remark 4.4.6.** By Proposition 4.4.1, the coordinate ring \( R \) for a generic collection of lines is not Cohen-Macaulay, but \( \operatorname{reg}_S(R) = \alpha \), where \( \alpha \) is precisely the smallest non-negative integer where
Hilb\(d\) = HilbP\(d\) for \(d \geq \alpha\). By Proposition 4.3.2, if a ring is Cohen-Macaulay then the regularity is precisely this number. So, even though we are not Cohen-Macaulay, we do not lose everything in generalizing these theorems.

Compare the previous result with the following general regularity bound for intersections of ideals generated by linear forms.

**Theorem 4.4.7. (Derksen and Sidman (2002))** If \( J = \bigcap_{i=1}^{j} I_i \) is an ideal of \( S \), where each \( I_i \) is an ideal generated by linear forms, then \( \text{reg}_S(S/J) \leq j \).

The assumption that \( R \) is a coordinate ring of a generic collection of lines tells us the regularity exactly, which is much smaller than the Derksen-Sidman bound for a fixed \( n \). By way of comparison we compute the following estimate.

**Corollary 4.4.8.** Let \( M \) be a generic collection of \( m \) lines in \( \mathbb{P}^n \) with \( n \geq 3 \) and \( R \) the coordinate ring of \( M \). Then

\[
\text{reg}_S(R) \leq \left\lceil \sqrt[n]{n!} \left( \sqrt[n]{m} - 1 \right) \right\rceil.
\]

**Proof.** Let \( p(x) = (x + n) \cdots (x + 2) - n!m \). The polynomial \( p(x) \) has a unique positive root by the Intermediate Value Theorem, since the \((x + n) \cdots (x + 2)\) is increasing on the non-negative real numbers. Let \( a \) be this positive root, and observe that the smallest non-negative integer \( \alpha \) satisfying the inequality \( \left( \frac{n + \alpha}{\alpha} \right) \geq m(\alpha + 1) \) is precisely the ceiling of the root \( a \).

We now use an inequality of Minkowski (Frenkel and Horváth (2014)). If \( x_k \) and \( y_k \) are positive for each \( k \), then

\[
\sqrt[n-1]{\prod_{k=1}^{n-1} (x_k + y_k)} \geq \sqrt[n-1]{\prod_{k=1}^{n-1} x_k} + \sqrt[n-1]{\prod_{k=1}^{n-1} y_k}.
\]

Thus,

\[
\sqrt[n]{n!} m = \sqrt[n]{(a + n) \cdots (a + 2)}
\]

\[
\geq a + \sqrt[n]{n!}
\]
Therefore,
\[
\sqrt[n]{n!m} - \sqrt[n]{n!} \geq a.
\]
Taking ceilings gives the inequality.

We would like to note that \( \text{reg}_S(R) \) is roughly asymptotic to the upper bound. Proposition 4.4.1 and Theorem 4.4.5 tell us the coordinate ring \( R \) of a non-trivial generic collection of lines in \( \mathbb{P}^n \) is not Cohen-Macaulay, \( \text{pdim}_S(R) = n \), and the regularity is the smallest non-negative integer \( \alpha \) satisfying \( \binom{n+\alpha}{\alpha} \geq m(\alpha + 1) \). So, the resolution of \( R \) is well-behaved, in the sense that if \( n \) is fixed and we allow \( m \) to vary we may expect the regularity to be low compared to the number of lines in our collection.

### 4.5 Koszul Filtrations for Collections of Lines

In this section we determine when a generic collection of lines, or a collection of lines in general linear position, will yield a Koszul coordinate ring. To this end, most of the work will be in constructing a Koszul filtration in the coordinate ring of a generic collection of lines.

**Proposition 4.5.1.** Let \( \mathcal{M} \) be a collection of \( m \) lines in general linear position in \( \mathbb{P}^n \), with \( n \geq 3 \) and \( R \) the coordinate ring of \( \mathcal{M} \). If \( n + 1 \geq 2m \), then after a change of basis the defining ideal is minimally generated by monomials of degree at most 2. Thus, \( R \) is Koszul.

**Proof.** We use \( \hat{\cdot} \) to denote a term removed from a sequence. Let \( R \) be the coordinate ring of \( \mathcal{M} \) with defining ideal \( J \). Through a change of basis and Remark 4.3.8 we may assume the defining ideal for each line has the following form

\[
L_i = (x_0, \ldots, \hat{x}_{n-2i+1}, \hat{x}_{n-2i+2}, \ldots, x_{n-1}, x_n),
\]

for \( i = 1, \ldots, m \). Since every \( L_i \) is monomial, so is \( J \). Furthermore, since \( n + 1 \geq 2m \), the \( \text{reg}_S(R) \leq 1 \). Thus, \( J \) is generated by monomials of degree at most 2. Theorem 4.3.4 guarantees \( R \) is Koszul. \( \square \)
Un fortunately, the simplicity of the previous proof does not carry over for larger generic collections of lines. We need a lemma.

**Lemma 4.5.2.** Let $\mathcal{M}$ be a generic collection of $m$ lines in $\mathbb{P}^n$ and $R$ the coordinate ring of $\mathcal{M}$. If $\text{reg}_S(R) = 1$, then the Hilbert series of $R$ is

$$H_{S/J}(t) = \frac{(1 - m)t^2 + (m - 2)t + 1}{(1 - t)^2}.$$

If $\text{reg}_S(R) = 2$, then the Hilbert series of $R$ is

$$H_{S/J}(t) = \frac{(1 + n - 2m)t^3 + (3m - 2n - 1)t^2 + (n - 1)t + 1}{(1 - t)^2}.$$

**Proof.** By Theorem 4.4.5, the regularity is the smallest non-negative integer $\alpha$ satisfying $\binom{n + \alpha}{\alpha} \geq m(\alpha + 1)$. Suppose $\text{reg}_S(R) = 1$. By Theorem 4.4.4, the Hilbert series for $R$ is

$$H_R(t) = 1 + 2mt + 3mt^2 + 4mt^3 + \cdots$$

$$= 1 - m \left( \frac{t(t - 2)}{(1 - t)^2} \right)$$

$$= \frac{t^2 - 2t + 1 - mt^2 + 2mt}{(1 - t)^2}$$

$$= \frac{(1 - m)t^2 + 2(m - 1)t + 1}{(1 - t)^2}.$$

Now, suppose $\text{reg}_S(R) = 2$. By Theorem 4.4.4, the Hilbert series for $R$ is

$$H_R(t) = 1 + (n + 1)t + 3mt^2 + 4mt^3 + \cdots$$

$$= 1 + (n + 1)t - m \left( \frac{t^2(2t - 3)}{(1 - t)^2} \right)$$

$$= \frac{(n + 1)t^3 - (2n + 1)t^2 + (n - 1)t + 1 - 2mt^3 + 3mt^2}{(1 - t)^2}$$

$$= \frac{(n + 1 - 2m)t^3 + (3m - 2n - 1)t^2 + (n - 1)t + 1}{(1 - t)^2}.$$

We can now construct a Koszul filtration for the coordinate ring of certain larger generic collections of lines.
Theorem 4.5.3. Let $\mathcal{M}$ be a generic collection of $m$ lines in $\mathbb{P}^n$ such that $n \geq 3$ and $m \geq 3$ and $R$ the coordinate ring of $\mathcal{M}$.

(a) If $m$ is even and $m + 1 \leq n$, then $R$ has a Koszul filtration.

(b) If $m$ is odd and $m + 2 \leq n$, then $R$ has a Koszul filtration.

In particular, $R$ is Koszul in these cases.

Proof. We only prove (a) due to the length of the proof and note that (b) is done identically except for the Hilbert series computations. In both cases we may assume that $n \leq 2(m - 1)$, otherwise Proposition 4.5.1 and Remark 4.3.8 prove the claim. By Remark 4.3.8 and a change of basis, we may assume the defining ideals for our $m$ lines have the following form

\begin{align*}
L_1 &= (x_0, \ldots, x_{n-4}, x_{n-3}, x_{n-2}) \\
L_2 &= (x_0, \ldots, x_{n-4}, x_{n-1}, x_n) \\
& \vdots \\
L_i &= (x_0, \ldots, \hat{x}_{n-2i+1}, \hat{x}_{n-2i+2}, \ldots, x_n) \\
& \vdots \\
L_k &= (x_0, \ldots, \hat{x}_{n-2k+1}, \hat{x}_{n-2k+2}, \ldots, x_n) \\
L_{k+1} &= (l_0, \ldots, l_{n-4}, l_{n-3}, l_{n-2}) \\
L_{k+2} &= (l_0, \ldots, l_{n-4}, l_{n-1}, l_n) \\
& \vdots \\
L_{k+i} &= (l_0, \ldots, \hat{l}_{n-2i+1}, \hat{l}_{n-2i+2}, \ldots, l_n) \\
& \vdots \\
L_{2k} &= (l_0, \ldots, \hat{l}_{n-2k+1}, \hat{l}_{n-2k+2}, \ldots, l_n),
\end{align*}

where $l_i$ are general linear forms in $S$. Denote the ideals

\begin{align*}
J &= \bigcap_{i=1}^{2k} L_i,
K &= \bigcap_{i=1}^{k} L_i,
I &= \bigcap_{i=k+1}^{2k} L_i,
\end{align*}

where $l_i$ are general linear forms in $S$. Denote the ideals

so that \( J = K \cap I \). Let \( R = S/J \); to prove that \( R \) is Koszul we will construct a Koszul filtration. To construct the filtration we need the two Hilbert series \( H_{(J + (x_0)):(x_1)}(t) \) and \( H_{(J + (l_0)):(l_1)}(t) \). We first calculate the former. Observe \( (x_0, x_1) \subseteq L_i \) and \( (l_0, l_1) \subseteq L_{k+i} \) for \( i = 1, \ldots, k \). Using the modular law (Atiyah and Macdonald (2016)), we have the equality

\[
(J + (x_0)) : (x_1) = (K \cap I + K \cap (x_0)) : (x_1) = (I + (x_0)) : (x_1).
\]

So, it suffices to determine \( H_{S/J + (x_0)}(t) \). To this end, we first calculate \( H_{S/(I + (x_0, x_1))}(t) \). To do so we use the short exact sequence

\[
0 \to S/(I + (x_0)) \cap (I + (x_1)) \to S/(I + (x_0)) \oplus S/(I + (x_1)) \to S/(I + (x_0, x_1)) \to 0.
\]

Our assumption \( m + 1 \leq n \leq 2(m - 1) \) guarantees that \( \operatorname{reg}_S(S/I) = 1 \). Thus, by Lemma 4.5.2

\[
H_{S/I}(t) = \frac{(1 - k)t^2 + 2(k - 1)t + 1}{(1 - t)^2},
\]

and since \( x_0 \) and \( x_1 \) are nonzerodivisors on \( S/I \), we have the following two Hilbert series

\[
H_{S/(I + (x_0))}(t) = H_{S/(I + (x_1))}(t) = \frac{(1 - k)t^2 + 2(k - 1)t + 1}{1 - t}.
\]

Furthermore, the coordinate ring \( S/(I + (x_0)) \cap (I + (x_1)) \) corresponds precisely to a collection of \( 2k \) distinct points. These points must necessarily be in general linear position, since by assumption \( 2k = m \leq n + 1 \) and no three are collinear. So, by Theorem 4.4.3 we have

\[
H_{S/((I + (x_0)) \cap (I + (x_1)))}(t) = \frac{(2k - 1)t + 1}{1 - t}.
\]

By the additivity of the Hilbert series

\[
H_{S/(I + (x_0, x_1))}(t) = H_{S/(I + (x_0))}(t) + H_{S/(I + (x_1))}(t) - H_{S/(I + (x_0)) \cap (I + (x_1))}(t)
= 2 \left( \frac{(1 - k)t^2 + 2(k - 2)t + 1}{1 - t} \right) - \frac{(2k - 1)t + 1}{1 - t}
= 1 + 2(k - 1)t.
\]
Thus, by the short exact sequence

\[ 0 \to S/((I + (x_0)) : (x_1)) (-1) \to S/(J + (x_0)) \to S/(I + (x_0, x_1)) \to 0, \]

Equation (4.3), and the additivity of the Hilbert series

\[ H_{S/(J+(x_0)): (x_1)}(t) = H_{S/I+(x_0) : (x_1)}(t) \]

\[ = \frac{1}{t} \left( (1-k)t^2 + 2(k-1)t + 1 - 1 - 2(k-1)t \right) \]

\[ = \frac{(k-1)t + 1}{1 - t}. \]

This gives us our desired Hilbert series. An identical argument and interchanging \( I \) with \( K \) and \( x_0 \) and \( x_1 \) with \( l_0 \) and \( l_1 \) yields

\[ H_{S/K}(t) = \frac{(1-k)t^2 + 2(k-1)t + 1}{(1-t)^2}, \]  

\[ H_{S/(K+(l_0))}(t) = H_{S/(K+(l_1))}(t) = \frac{(1-k)t^2 + 2(k-1)t + 1}{(1-t)^2}, \]

and

\[ H_{S/(J+(l_0)) : (l_1)}(t) = H_{S/(J+(x_0)) : (x_1)}(t). \]

We can now define a Koszul filtration \( \mathcal{F} \) for \( R \). We use \( \bar{\cdot} \) to denote the image of an element of \( S \) in \( R = S/J \) for the remainder of the paper. We have already seen in Equation (4.5) that

\[ H_{S/(J+(x_0)) : (x_1)}(t) = \frac{(k-1)t + 1}{1 - t} = 1 + \sum_{i=1}^{\infty} kt^i. \]

Hence, \( n-k+1 \) linearly independent linear forms are in a minimal generating set of \( (J+(x_0)) : (x_1) \).

Clearly \( l_0, \ldots, l_{n-2k}, x_0 \in (J+(x_0)) : (x_1) \), label \( z_{n-2k+2}, \ldots, z_{n-k} \) as the remaining linear forms from a minimal generating set of \( (J+(x_0)) : (x_1) \). Similarly, choose \( y_i \) from \( (J+(l_0)) : (l_1) \) so that \( x_0, x_1, \ldots, x_{n-2k}, l_0, y_{n-k+2}, \ldots, y_{n-k} \) are linear forms forming a minimal generating set of \( (J+(l_0)) : (l_1) \).

The set \( \{ l_0, \ldots, l_{n-2k}, x_0, z_{n-2k+1}, \ldots, z_{n-k}, x_1 \} \) is a linearly independent set over \( S \), otherwise \( x_1^2 \in J+(x_0) \). This means \( x_1^2 \in (L_i + (x_0)) \) for \( i = k + 1, \ldots, 2k \), a contradiction. Similarly
\[\{x_0, \ldots, x_{n-2k}, l_0, y_{n-2k}, \ldots, y_{n-k}, l_1\}\] is linearly independent over \(S\). Let \(w_{n-k+2}, \ldots, w_{n+1}\) and \(u_{n-k+2}, \ldots, u_{n+1}\) be extensions of

\[\{\bar{l}_0, \ldots, \bar{l}_{n-2k}, \bar{x}_0, \bar{z}_{n-2k}, \ldots, \bar{z}_{n-k}, \bar{x}_1\}\]

and

\[\{\bar{x}_0, \ldots, \bar{x}_{n-2k}, \bar{l}_0, \bar{y}_{n-2k}, \ldots, \bar{y}_{n-k}, \bar{l}_1\}\]

to minimal systems of generators of \(m_R\), respectively. Define \(\mathcal{F}\) as follows

\[
\mathcal{F} = \begin{cases}
  0, (\bar{x}_0), (\bar{x}_0, \bar{x}_1), (l_0, \bar{l}_0, \bar{l}_1), \\
  \vdots \\
  (\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{n-2k}), (l_0, \bar{l}_1, \bar{l}_n, \ldots, \bar{l}_{n-k}), \\
  (\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{n-2k}, \bar{l}_0, \bar{y}_{n-2k}, \ldots, \bar{y}_{n-k}, \bar{l}_1), \\
  (l_0, \bar{l}_1, \bar{l}_n, \bar{x}_0, \bar{z}_{n-2k}, \ldots, \bar{z}_{n-k}, \bar{x}_1), \\
  \vdots \\
  (\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{n-2k}, \bar{l}_0, \bar{y}_{n-2k}, \ldots, \bar{y}_{n-k}, \bar{l}_1, u_{n-k+2}), \\
  (l_0, \bar{l}_1, \bar{l}_n, \bar{x}_0, \bar{z}_{n-2k}, \ldots, \bar{z}_{n-k}, \bar{x}_1, w_{n-k+2}), \\
  \vdots \\
  (\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{n-2k}, \bar{l}_0, \bar{y}_{n-2k}, \ldots, \bar{y}_{n-k}, \bar{l}_1, u_{n-k+2}, \ldots, u_n), \\
  (l_0, \bar{l}_1, \bar{l}_n, \bar{x}_0, \bar{z}_{n-2k}, \ldots, \bar{z}_{n-k}, \bar{x}_1, w_{n-k+2}, \ldots, w_n), \\
  m_R
\end{cases}
\]
We now prove $F$ is a Koszul filtration. We do this by proving several claims. Throughout the process we use the inclusion $(x_0, x_1) \cap (l_0, l_1) \subseteq J$. Afterwards, we summarize all computed colons and list the claims that prove the calculated colons.

Claim 4.5.4. The ideal $(\bar{x}_0, \bar{x}_1, \bar{l}_0, \bar{l}_1)$ in $R$ has Hilbert series

$$H_{R/(\bar{x}_0, \bar{x}_1, \bar{l}_0, \bar{l}_1)}(t) = 1 + (n - 3)t$$

and any ideal $P$ containing this ideal has the property that $P : (\ell) = m_R$, where $\ell$ is a linear form not contained in $P$.

Proof. We begin by observing that our assumption $m + 1 \leq n \leq 2(m - 1)$ and Proposition 4.4.5 yield $\text{reg}_S(S/J) = 2$. Thus, by Lemma 4.5.2

$$H_{S/J}(t) = \frac{(n + 1 - 4k)t^3 + (6k - 2n - 1)t^2 + (n - 1)t + 1}{(1 - t)^2}.$$  

Now, $L_i : (x_0) = L_i$ for $i = k + 1, \ldots, 2k$, since $x_0 \notin L_i$. Thus,

$$J : (x_0) = \left( \bigcap_{i=1}^{2k} L_i \right) : (x_0) = \bigcap_{i=1}^{2k} (L_i : (x_0)) = \bigcap_{i=k+1}^{2k} L_i = I.$$  

So, $H_{S/(J : (x_0))}(t) = H_{S/I}(t)$. Using the short exact sequence

$$0 \rightarrow S/(J : (x_0))(-1) \rightarrow S/J \rightarrow S/(J + (x_0)) \rightarrow 0,$$

Equation (4.4), and the additivity of the Hilbert series yields

$$H_{S/(J + (x_0))}(t) = H_{S/J}(t) - tH_{S/(J : (x_0))}(t)$$

$$= H_{S/J}(t) - tH_{S/I}(t)$$

$$= \frac{(1+n-4k)t^3 + (6k - 2n - 1)t^2 + (n - 1)t + 1}{(1 - t)^2} - t \left( \frac{(1-k)t^2 + 2(k-1)t + 1}{(1 - t)^2} \right)$$

$$= \frac{(n - 3k)t^3 + (4k - 2n + 1)t^2 + (n - 2)t + 1}{(1 - t)^2}.$$  

Using the short exact sequence

$$0 \rightarrow S/((J + (x_0)) : (x_1))(-1) \rightarrow S/(J + (x_0)) \rightarrow S/(J + (x_0, x_1)) \rightarrow 0,$$
Equation (4.5), the previous Hilbert series, and the additivity of the Hilbert series yields

\[
H_{S/(J+(x_0,x_1))}(t) = H_{S/(J+(x_0))}(t) - tH_{S/(J+(x_0))}(x_1)(t) \\
= \frac{(n-3k)t^3+(4k-2n+1)t^2+(n-2)t+1}{(1-t)^2} - t \left( \frac{(k-1)t+1}{1-t} \right) \\
= \frac{(n-2k-1)t^3+(3k-2n+3)t^2+(n-3)t+1}{(1-t)^2}.
\]

Replacing \(x_0\) and \(x_1\) with \(l_0\) and \(l_1\) demonstrates that

\[
H_{S/(J+(x_0,x_1))}(t) = H_{S/(J+(l_0,l_1))}(t).
\]

Thus, using the short exact sequence

\[
0 \to R/((x_0, x_1) \cap (l_0, l_1)) \to R/(x_0, x_1) \oplus R/(l_0, l_1) \to R/(\bar{x}_0, \bar{x}_1, \bar{l}_0, \bar{l}_1) \to 0,
\]

and the additivity of the Hilbert series yields

\[
H_{R/(\bar{x}_0, \bar{x}_1, \bar{l}_0, \bar{l}_1)}(t) = H_{R/(\bar{x}_0, \bar{x}_1)}(t) + H_{R/(\bar{l}_0, \bar{l}_1)}(t) - H_R(t) \\
= 2 \left( \frac{(n-2k-1)t^3+(3k-2n+3)t^2+(n-3)t+1}{(1-t)^2} \right) \\
- \frac{(1+n-4k)t^3+(6k-2n-1)t^2+(n-1)t+1}{(1-t)^2} \\
= \frac{(n-3)t^3+(7-2n)t^2+(n-5)t+1}{(1-t)^2} \\
= 1 + (n-3)t.
\]

So, \(R_2 \subset (\bar{x}_0, \bar{x}_1, \bar{l}_0, \bar{l}_1)\). This means that any ideal \(P \subset R\) containing the ideal \((\bar{x}_0, \bar{x}_1, \bar{l}_0, \bar{l}_1)\) has the property that \(P : (\ell) = \mathfrak{m}_R\), where \(\ell\) is a linear form not contained in \(P\). \(\square\)

**Claim 4.5.5.** For \(i = 1, \ldots, n-2k\), we have the two Hilbert series

\[
H_{R/(\bar{x}_0, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_i)}(t) = H_{R/(l_0, l_1, l_2, \ldots, l_i)}(t) \\
= \frac{(n-2k-i)t^3+(3k-2n+2i+1)t^2+(n-(i+2))t+1}{(1-t)^2},
\]

and the two equalities \(J + (x_0, \ldots, x_{n-2k}) = K\), and \(J + (l_0, \ldots, l_{n-2k}) = I\).
Proof. Adding the linear forms $x_2, \ldots, x_i$ to the ideal $(\bar{x}_0, \bar{x}_1, \bar{l}_0, \bar{l}_1)$ yields

$$H_R/(\bar{x}_0, \bar{x}_1, \bar{l}_0, \bar{l}_1, x_2, \ldots, x_i)(t) = H_R/(\bar{x}_0, \bar{x}_1, \bar{l}_0, \bar{l}_1, \bar{x}_2, \ldots, \bar{x}_i)(t) = 1 + (n - (i + 2))t$$

for $i = 2, \ldots, n - 2k$. Using the short exact sequence

$$0 \to R/((\bar{x}_0, \bar{x}_1, \bar{x}_2) \cap (\bar{l}_0, \bar{l}_1)) \to R/(\bar{x}_0, \bar{x}_1, \bar{x}_2) \oplus R/(\bar{l}_0, \bar{l}_1) \to R/(\bar{x}_0, \bar{x}_1, \bar{l}_0, \bar{l}_1) \to 0$$

and the additivity of the Hilbert series gives

$$H_R/(\bar{x}_0, \bar{x}_1, \bar{x}_2)(t) = H_R/(\bar{x}_0, \bar{x}_1, \bar{l}_0, \bar{l}_1, \bar{x}_2)(t) + H_R(t) - H_R/(\bar{l}_0, \bar{l}_1)(t)$$

$$= 1 + (n - 4)t + \frac{(1+n-4k)t^3+(6k-2n-1)t^2+(n-1)t+1}{(1-t)^2}$$

$$- \frac{(n-2k-1)t^3+(3k-2n+3)t^2+(n-3)t+1}{(1-t)^2}$$

$$= 1 + (n - 4)t + \frac{(2 - 2k)t^3 + (3k - 4)t^2 + 2t}{(1-t)^2}$$

$$= \frac{(n - 2k - 2)t^3 + (3k - 2n + 5)t^2 + (n - 4)t + 1}{(1-t)^2}.$$  

Replacing $(\bar{x}_0, \bar{x}_1, \bar{x}_2)$ with $(\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the above short exact sequence and using the additivity of the Hilbert series yields

$$H_R/(\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3)(t) = H_R/(\bar{x}_0, \bar{x}_1, \bar{l}_0, \bar{l}_1, \bar{x}_2, \bar{x}_3)(t) + H_R(t) - H_R/(\bar{l}_0, \bar{l}_1)(t)$$

$$= 1 + (n - 5)t + \frac{(1+n-4k)t^3+(6k-2n-1)t^2+(n-1)t+1}{(1-t)^2}$$

$$- \frac{(n-2k-1)t^3+(3k-2n+3)t^2+(n-3)t+1}{(1-t)^2}$$

$$= 1 + (n - 5)t + \frac{(2 - 2k)t^3 + (3k - 4)t^2 + 2t}{(1-t)^2}$$

$$= \frac{(n - 2k - 3)t^3 + (3k - 2n + 7)t^2 + (n - 5)t + 1}{(1-t)^2}.$$  

By induction

$$H_R/(\bar{x}_0, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_i)(t) = \frac{(n-2k-i)t^3+(3k-2n+2i+1)t^2+(n-(i+2))t+1}{(1-t)^2}$$  \hspace{1cm} (4.7)
for $i = 2, \ldots, n - 2k$. Setting $i = n - 2k$ we obtain the Hilbert series

$$H_{R/(x_0, \ldots, \bar{x}_{n-2k})}(t) = H_{R/(x_0, \bar{x}_1, \bar{\ell}_0, \bar{x}_2, \ldots, \bar{x}_{n-2k})}(t)$$

$$+ H_{R}(t) - H_{R/(\bar{\ell}_0, \bar{\ell}_1)}(t)$$

$$= 1 + (2k - 2)t + \frac{(1+n-4k)t^2 + (6k-2n-1)t^3 + (n-1)t + 1}{(1-t)^2}$$

$$- \frac{(n-2k-1)t^3 + (3k-2n+3)t^2 + (n-3)t + 1}{(1-t)^2}$$

$$= 1 + (2k - 2)t + \frac{(2 - 2k)t^3 + (3k - 4)t^2 + 2t}{(1-t)^2}$$

$$= \frac{(1 - k)t^2 + 2(k - 1)t + 1}{(1-t)^2}.$$

Interchanging each $x_i$ with $l_i$ gives us the other desired Hilbert series.

The Hilbert series in Equation (4.7) is the same as in Equation (4.6). Furthermore $J + (x_0, \ldots, x_{n-2k}) \subseteq K$. So, we have that $J + (x_0, \ldots, x_{n-2k}) = K$ and interchanging each $x_i$ with $l_i$ gives us the other equality. \qed

**Claim 4.5.6.** We have the equalities

$$(\bar{\ell}_0, \ldots, \bar{\ell}_{n-2k}, \bar{x}_0) : (\bar{z}_{n-2k+2}) = m_R,$$

$$(\bar{x}_0, \ldots, \bar{x}_{n-2k}, \bar{\ell}_0) : (\bar{y}_{n-2k+2}) = m_R,$$

and

$$(\bar{\ell}_0, \ldots, \bar{\ell}_{n-2k}, \bar{x}_0, \bar{z}_{n-2k+2}, \ldots, \bar{z}_{i+1}) : (\bar{z}_{i+1}) = m_R,$$

$$(\bar{x}_0, \ldots, \bar{x}_{n-2k}, \bar{\ell}_0, \bar{y}_{n-2k+2}, \ldots, \bar{y}_{i+1}) : (\bar{y}_{i+1}) = m_R,$$

for $i = n - 2k + 2, \ldots, n - k$. Furthermore,

$$(\bar{x}_0) : (\bar{x}_1) = (\bar{\ell}_0, \ldots, \bar{\ell}_{n-2k}, \bar{x}_0, \bar{z}_{n-2k+2}, \ldots, \bar{z}_{n-k})$$

and

$$(\bar{\ell}_0) : (\bar{\ell}_1) = (\bar{x}_0, \ldots, \bar{x}_{n-2k}, \bar{\ell}_0, \bar{y}_{n-2k+2}, \ldots, \bar{y}_{n-k}).$$
Proof. We begin by observing

\[(\bar{l}_0, \bar{l}_1, \bar{x}_0, \bar{x}_1) \subseteq (\bar{l}_0, \bar{l}_1, \bar{x}_0, \bar{x}_2, \ldots, \bar{l}_{n-2k}) : (\bar{x}_n-2k+2).\]

So by Claim 4.5.4, we conclude that

\[H_{R/((\bar{l}_0, \bar{l}_1, \ldots, \bar{l}_{n-2k}, \bar{x}_0):(\bar{x}_n-2k+2))}(t) = 1 + \alpha t,\]

where \(\alpha \in \{0, 1, \ldots, n-3\}\). Using the short exact sequence

\[0 \rightarrow R/((\bar{x}_0, \bar{l}_0, \ldots, \bar{l}_{n-2k}) : (\bar{x}_n-2k+2)) \rightarrow R/(\bar{x}_0, \bar{l}_0, \ldots, \bar{l}_{n-2k}) \rightarrow R/(\bar{x}_0, \bar{l}_0, \ldots, \bar{l}_{n-2k}, \bar{z}_n-2k+2) \rightarrow 0,\]

Claim 4.5.5, and that the fact that \(x_0\) is a nonzerodivisor on \(S/I\) we obtain

\[H_{R/((\bar{x}_0, \bar{l}_0, \ldots, \bar{l}_{n-2k}, \bar{z}_n-2k+2))}(t) = H_{R/((\bar{x}_0, \bar{l}_0, \ldots, \bar{l}_{n-2k}))}(t) - tH_{R/((\bar{x}_0, \bar{l}_0, \ldots, \bar{l}_{n-2k}))}(t) = \frac{(1-k)t^2 + 2(k-1)t + 1}{(1-t)} - t(1 + \alpha t)\]

\[= \frac{\alpha t^3 + (2 - \alpha - k)t^2 + (2k - 3)t + 1}{(1-t)}\]

\[= 1 + (2k - 2)t + (k - \alpha)t^2 + \sum_{j=3}^{\infty} kt^j.\]

We also have the containment

\[(\bar{l}_0, \ldots, \bar{l}_{n-2k}, \bar{x}_0) \subseteq (\bar{l}_0, \ldots, \bar{l}_{n-2k}, \bar{x}_0) : (\bar{x}_1).\]

Using Claim 4.5.5 we obtain the equality

\[(J + (l_0, \ldots, l_{n-2k}, x_0)) : (x_1) = (I + (x_0)) : (x_1),\]

which has Hilbert series computed in (4.5). Hence

\[H_{R/((l_0, l_1, \ldots, l_{n-2k}, x_0):(x_1))}(t) = \frac{(k-1)t + 1}{1-t} = 1 + \sum_{j=1}^{\infty} kt^j.\]
Comparing coefficients of \( t^2 \) yields \( k \leq k - \alpha \), and so \( \alpha = 0 \), proving the first equality. We immediately have the equality

\[
(\bar{l}_0, \ldots, \bar{l}_{n-2k}, \bar{x}_0, \bar{z}_{n-2k+2}, \ldots, \bar{z}_i) : (\bar{z}_{i+1}) = \mathfrak{m}_R,
\]

for each \( i = n - 2k + 2, \ldots, n - k \).

Notice that setting \( \alpha = 0 \), yields

\[
H_{\mathcal{R}/(\bar{l}_0, \ldots, \bar{l}_{n-2k}, \bar{x}_0, \bar{z}_{n-2k+2})}(t) = \frac{(2-k)t^2 + (2k-3)t + 1}{(1-t)}.
\]

Denote \( V_i \) and \( V'_i \) to be the ideals

\[
V_i = (\bar{l}_0, \ldots, \bar{l}_{n-2k}, \bar{x}_0, \bar{z}_{n-2k+2}, \ldots, \bar{z}_i)
\]

and

\[
V'_i = (\bar{x}_0, \ldots, \bar{x}_{n-2k}, \bar{l}_0, \bar{y}_{n-2k+2}, \ldots, \bar{y}_i)
\]

for \( i = n - 2k + 2, \ldots, n - k \). Replacing the ideals in (4.8) with the three ideals \( V_{n-2k+3}, V_{n-2k+2}, \) and \( V_{n-2k+2} : (\bar{z}_{n-2k+3}) \), and using the additivity of the Hilbert series yields

\[
H_{\mathcal{R}/V_{n-2k+3}}(t) = H_{\mathcal{R}/V_{n-2k+2}}(t) - tH_{\mathcal{R}/V_{n-2k+2} : (\bar{z}_{n-2k+3})}(t) = \frac{(2-k)t^2 + (2k-3)t + 1}{(1-t)} - t
\]

\[
= \frac{(3-k)t^2 + (2k-4)t + 1}{(1-t)}.
\]

Continuing in this fashion gives

\[
H_{\mathcal{R}/V_{n-k}}(t) = \frac{(k-1)t + 1}{(1-t)}.
\]

So, both \( (\bar{x}_0) : (\bar{x}_1) \) and \( V_{n-k} \) have the same Hilbert series. Furthermore, \( V_{n-k} \subseteq (\bar{x}_0) : (\bar{x}_1) \). So these ideals are in fact equal. Interchanging \( x_0 \) and \( x_1 \) with \( l_0 \) and \( l_1 \) yields the remaining equality. \( \Box \)
Claim 4.5.7. We have the equalities

\[(0_R) : (\bar{x}_0) = (\bar{l}_0, \ldots, \bar{l}_{n-2k})\]

and

\[(0_R) : (\bar{l}_0) = (\bar{x}_0, \ldots, \bar{x}_{n-2k}).\]

Proof. The two equalities follow immediately since \((0_R) : (\bar{x}_0) \subseteq (\bar{l}_0, \ldots, \bar{l}_{n-2k})\) and \((0_R) : (\bar{l}_0) \subseteq (\bar{x}_0, \ldots, \bar{x}_{n-2k})\) and all four ideals have the same Hilbert series by Claim 4.5.5.

Claim 4.5.8. We have the equality

\[(\bar{x}_0, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_i) : (\bar{x}_{i+1}) = m_R,\]

for \(i = 2, \ldots, n - 2k - 1\).

Proof. Using the short exact sequence

\[0 \to R/(\bar{x}_0, \ldots, \bar{x}_i) : (\bar{x}_{i+1})(-1) \to R/(\bar{x}_0, \ldots, \bar{x}_i) \to R/(\bar{x}_0, \ldots, \bar{x}_i, \bar{x}_{i+1}) \to 0,\]

and the Hilbert series from (4.7), we get

\[H_{R/(\bar{x}_0, \ldots, \bar{x}_i) : (\bar{x}_{i+1})}(t) = \frac{1}{t} \left( H_{R/(\bar{x}_0, \ldots, \bar{x}_i)}(t) - H_{R/(\bar{x}_0, \ldots, \bar{x}_i, \bar{x}_{i+1})}(t) \right)\]

\[= \frac{1}{t} \left( \frac{(n-2k-i)t^3+(3k-2n+2i+1)t^2+(n-(i+2))t+1}{(1-t)^2} - \frac{(n-2k-i-1)t^3+(3k-2n+2i+3)t^2+(n-(i+3))t+1}{(1-t)^2} \right)\]

\[= \frac{t^2 - 2t + 1}{(1 - t)^2} = 1,\]

proving the claim.

Claim 4.5.9. We have the four equalities

\[(\bar{x}_0, \ldots, \bar{x}_{n-2k}) : (\bar{l}_0) = (\bar{x}_0, \ldots, \bar{x}_{n-2k}),\]
\[(\bar{l}_0, \ldots, \bar{l}_{n-2k}) : (\bar{x}_0) = (\bar{l}_0, \ldots, \bar{l}_{n-2k}) ,\]

\[V'_{n-k} : (\bar{l}_1) = V'_{n-k} ,\]

\[V_{n-k} : (\bar{x}_1) = V_{n-k} .\]

**Proof.** The equality

\[(\bar{x}_0, \ldots, \bar{x}_{n-2k}) : (\bar{l}_0) = (\bar{x}_0, \ldots, \bar{x}_{n-2k})\]

follows from the genericity of \(l_0\). We now aim to show the equality

\[V_{n-k} : (\bar{x}_1) = V_{n-k} .\]

We always have the containment \(V_{n-k} \subseteq V_{n-k} : (\bar{x}_1)\) and by Claim 4.5.6 we have already determined \(H_{R/V_{n-k}}(t)\). So, we must only determine \(H_{R/(V_{n-k} : (\bar{x}_1))}(t)\). We aim to use the additivity of the Hilbert series along the short exact sequence

\[0 \to R/(V_{n-k} : (\bar{x}_1))(-1) \to R/V_{n-k} \to R/(V_{n-k} + (\bar{x}_1)) \to 0, \tag{4.10}\]

but we first must determine \(H_{R/(V_{n-k} + (\bar{x}_1))}(t)\). By Claim 4.5.4, adding the linear forms \(\bar{l}_2, \ldots, \bar{l}_{n-2k}, \bar{z}_{n-2k+2}, \ldots, \bar{z}_{n-k}\) to the ideal \((\bar{x}_0, \bar{x}_1, \bar{l}_0, \bar{l}_1)\) yields the Hilbert series

\[H_{R/(V_{n-k} + (\bar{x}_1))}(t) = 1 + (k - 1)t .\]

Using Claim 4.5.6 and the additivity of the Hilbert series along the short exact sequence (4.10) yields

\[H_{R/(V_{n-k} : (\bar{x}_1))}(t) = \frac{1}{t} \left( H_{R/V_{n-k}}(t) - H_{R/(V_{n-k} + (\bar{x}_1))}(t) \right) \]

\[= \frac{1}{t} \left( \frac{(k - 1)t + 1}{(1 - t)} - (1 + (k - 1)t) \right) \]

\[= \frac{(k - 1)t + 1}{(1 - t)} .\]

Interchanging \(\bar{x}_0\) and \(\bar{x}_1\) with \(\bar{l}_0\) and \(\bar{l}_1\), proves the other two equalities. \(\Box\)
Below is a list of calculated colons with the corresponding justification.

\[
\begin{align*}
(0) : (\bar{x}_0) &= (\bar{l}_0, \ldots, \bar{l}_{n-2k}), & (0) : (\bar{l}_0) &= (\bar{x}_0, \ldots, \bar{x}_{n-2k}), & 4.5.7 \\
(\bar{x}_0) : (\bar{x}_1) &= (\bar{l}_0, \ldots, \bar{l}_{n-2k}, \bar{x}_0, \bar{\bar{z}}_{n-2k+2}, \ldots, \bar{\bar{z}}_{n-k}), & 4.5.6 \\
(\bar{l}_0) : (\bar{l}_1) &= (\bar{x}_0, \ldots, \bar{x}_{n-2k}, \bar{l}_0, \bar{\bar{y}}_{n-2k+2}, \ldots, \bar{\bar{y}}_{n-k}), & 4.5.6 \\
(\bar{x}_0, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_i) : (\bar{x}_{i+1}) &= \mathfrak{m}_R, & i = 2, \ldots, n-2k-1, & 4.5.8 \\
(\bar{l}_0, \bar{l}_1, \bar{l}_2, \ldots, \bar{l}_i) : (\bar{l}_{i+1}) &= \mathfrak{m}_R, & i = 2, \ldots, n-2k-1, & 4.5.8 \\
(\bar{x}_0, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-2k}) : (\bar{l}_0) &= (\bar{l}_0, \ldots, \bar{l}_{n-2k}), & 4.5.9 \\
(\bar{l}_0, \bar{l}_1, \bar{l}_2, \ldots, \bar{l}_{n-2k}) : (\bar{x}_0) &= (\bar{x}_0, \ldots, \bar{x}_{n-2k}), & 4.5.9 \\
(\bar{x}_0, \ldots, \bar{x}_{n-2k}, \bar{l}_0) : (\bar{y}_{n-2k+2}) &= \mathfrak{m}_R, & 4.5.6 \\
(\bar{l}_0, \ldots, \bar{l}_{n-2k}, \bar{x}_0) : (\bar{z}_{n-2k+2}) &= \mathfrak{m}_R, & 4.5.6 \\
V'_{i} : (\bar{y}_{i+1}) &= \mathfrak{m}_R, & V_{i} : (\bar{z}_{i+1}) &= \mathfrak{m}_R, & i = n-2k+2, \ldots, n-k-1, & 4.5.6 \\
V'_{n-k} : (\bar{l}_1) &= V'_{n-k}, & V_{n-k} : (\bar{x}_1) &= V_{n-k}, & 4.5.9 \\
(V'_{n-k} + (\bar{l}_1)) : (\bar{u}_{n-k+1}) &= \mathfrak{m}_R, & (V'_{n-k} + (\bar{x}_1)) : (\bar{w}_{n-k+1}) &= \mathfrak{m}_R, & 4.5.4 \\
(V'_{n-k} + (\bar{l}_1, \bar{u}_{n-k+1}, \ldots, \bar{u}_i)) : (\bar{w}_{i+1}) &= \mathfrak{m}_R, & i = n-k+1, \ldots, n-1, & 4.5.4 \\
(V'_{n-k} + (\bar{x}_1, \bar{w}_{n-k+1}, \ldots, \bar{w}_i)) : (\bar{w}_{i+1}) &= \mathfrak{m}_R, & i = n-k+1, \ldots, n-1, & 4.5.4.
\end{align*}
\]

This completes the proof of Theorem 4.5.3. \qed

There is at least one example of a coordinate ring with the Koszul property which is not covered by our previous theorem. Let $\mathcal{M}$ be a generic collection of 5 lines in $\mathbb{P}^6$. By Remark 4.3.8 and a
change of basis we may assume the defining ideals for our 5 lines have the following form

\[ L_1 = (x_0, x_3, x_4, x_5, x_6) \quad L_2 = (x_0, x_1, x_4, x_5, x_2 + ax_3 + x_6) \]
\[ L_3 = (x_0, x_1, x_2, x_6, x_3 + bx_4 + x_5) \quad L_4 = (x_1, x_2, x_3, x_5, x_0 + 4 + x_6) \]
\[ L_5 = (x_2, x_3, x_4, x_6, x_0 + x_1 + x_5), \]

where \( a, b \in \mathbb{C} \) are algebraically independent over \( \mathbb{Q} \). Some further explanation is needed why we may assume our 5 lines have this form.

By Remark 4.3.8, the intersection of any triple of the defining ideals of our 5 lines contains a single linear form in a minimal generating set. Furthermore, the intersection of any pair of defining ideals for our 5 contains 3 linear forms in a minimal generating set. Thus, after a change of basis we may assume

\[ L_1 = (x_0, x_3, x_4, x_5, x_6) \quad L_2 = (x_0, x_1, x_4, x_5, l_0) \]
\[ L_3 = (x_0, x_1, x_2, x_6, l_1) \quad L_4 = (x_1, x_2, x_3, x_5, l_2) \]
\[ L_5 = (x_2, x_3, x_4, x_6, l_3). \]

where the linear forms \( l_0, \ldots, l_3 \) have the form

\[ l_0 = c_{0,2}x_2 + c_{0,3}x_3 + c_{0,6}x_6, \quad l_1 = c_{1,3}x_3 + c_{1,4}x_4 + c_{1,5}x_5, \]
\[ l_2 = c_{2,0}x_0 + c_{2,4}x_4 + c_{2,6}x_6, \quad l_3 = c_{3,0}x_0 + c_{3,1}x_1 + c_{3,5}x_5. \]

It is of no loss to assume these are all monic in certain indeterminates. That is they have the form

\[ l_0 = c_{0,2}x_2 + c_{0,3}x_3 + x_6, \quad l_1 = x_3 + c_{1,4}x_4 + c_{1,5}x_5, \]
\[ l_2 = x_0 + c_{2,4}x_4 + c_{2,6}x_6, \quad l_3 = c_{3,0}x_0 + c_{3,1}x_1 + x_5. \]

Through a change of basis we may reduce the coefficient on \( x_5 \) in \( l_1 \) to 1, and then normalize \( l_3 \) to be monic in \( x_5 \); then through another change of basis we may reduce the coefficient on \( x_0 \) in \( l_3 \) to 1, and then normalize \( l_2 \) to be monic in \( x_0 \); then through another change of basis we may reduce the coefficient on \( x_6 \) in \( l_2 \) to 1, and then normalize \( l_0 \) to be monic in \( x_6 \); then through another change
of basis we may reduce the coefficient on $x_2$ in $l_0$ to 1; then through another change of basis we may reduce the coefficient on $x_4$ in $l_2$ to 1; then through another change of basis we may reduce the coefficient on $x_1$ in $l_3$ to 1. Ultimately, we obtain

$$l_0 = x_2 + c_{0.3}x_3 + x_6,$$
$$l_1 = x_3 + c_{1.4}x_4 + x_5,$$
$$l_2 = x_0 + x_4 + x_6,$$
$$l_3 = x_0 + x_1 + x_5.$$

Note the order in which we make these reductions is important.

**Proposition 4.5.10.** Let $\mathcal{M}$ be a generic collection of 5 lines in $\mathbb{P}^6$ and $R$ the coordinate ring. Then $R$ is Koszul.

**Proof.** After a change of basis we may represent the defining ideal for our 5 lines as above. Below is a Koszul filtration

$$\mathcal{F} = \left\{ (0_R), (\bar{x}_0), (\bar{x}_2), (\bar{x}_0, \bar{x}_1), (\bar{x}_0, \bar{x}_3), (\bar{x}_0, \bar{x}_6), (\bar{x}_0, \bar{x}_2, \bar{x}_3), \ldots \right\}$$
\[
\begin{align*}
(x_0, x_1, x_3, x_4, x_5), (x_0, x_1, x_2, x_6, x_3 + b x_4), \\
(x_0, x_1, x_4, x_5, x_2 + a x_3 + x_6), (x_0, x_1, x_5, x_6, x_3 + b x_4), \\
(x_0, x_1, x_4, x_5, x_6), (x_0, x_1, x_2, x_5, x_6), (x_0, x_1, x_2, x_4, x_6), \\
(x_0, x_1, x_2, x_6, x_3 + b x_4 + x_5), (x_0, x_1, x_2, x_4, x_5, x_3 + \frac{1}{a} x_6), \\
(x_0, x_1, x_2, x_3, x_5, x_4 + x_6), (x_0, x_1, x_3, x_4, x_5, x_2 + x_6), \\
(x_0, x_1, x_3, x_4, x_5, x_6), (x_0, x_2, x_3, x_4, x_5, x_6), \\
(x_0, x_2, x_3, x_4, x_6, x_1 + x_5), (x_0, x_1, x_2, x_3, x_6, x_4 + \frac{1}{b} x_5) \\
(x_0, x_1, x_4, x_5, x_6, x_2 + a x_3 + x_6), (x_0, x_1, x_2, x_5, x_6, x_3 + b x_4), \\
(x_0, x_1, x_2, x_4, x_6, x_3 + x_5), m_R
\end{align*}
\]

The calculated colons are

\[
\begin{align*}
(0_R) : (x_0) &= (x_2, x_3, x_0 + x_1 + x_4 + x_5 + x_6), \\
(0_R) : (x_2) &= (x_0, x_4, x_5) \\
(\bar{x}_0) : (x_4) &= (x_0, x_1, x_2, x_3 + b \bar{x}_4 + \bar{x}_5 + b \bar{x}_6), \\
(\bar{x}_0) : (x_1) &= (x_0, x_3, x_4, x_6) \\
(\bar{x}_2) : (x_3) &= (x_0, x_1, x_2, x_3 + b \bar{x}_4 + \bar{x}_5 + \frac{1}{a} \bar{x}_6), \\
(\bar{x}_0) : (x_6) &= (x_0, x_1, x_5, x_2 + a \bar{x}_3 + x_4 + x_6) \\
(\bar{x}_2, \bar{x}_3) : (x_0) &= (x_2, x_3, x_0 + x_1 + \bar{x}_4 + x_5 + x_6) \\
(\bar{x}_2, \bar{x}_3) : (x_0 + x_1 + x_4 + x_5 + x_6) &= (x_0, x_2, x_3, x_6, x_4 + \frac{1}{b} x_5) \\
(\bar{x}_0, \bar{x}_4) : (x_5) &= (x_0, x_2, x_4, x_6, x_1 + x_3 + x_5) \\
(\bar{x}_0, \bar{x}_1) : (x_2) &= (x_0, x_1, x_4, x_5)(\bar{x}_0, x_4) : (x_3) = (\bar{x}_0, x_1, x_4, x_2 + a \bar{x}_3 + a \bar{x}_5 + \bar{x}_6) \\
(\bar{x}_0, \bar{x}_1) : (x_4) &= (x_0, x_1, x_2, x_3 + b \bar{x}_4 + \bar{x}_5 + b \bar{x}_6) \\
(\bar{x}_0, \bar{x}_4) : (x_2) &= (x_0, x_4, x_5)
\end{align*}
\]
\[(x_0, x_6) : (x_2) = (x_0, x_4, x_5, x_6)(x_0, x_1) : (x_5) = (x_0, x_1, x_2, x_6, x_3 + b x_4 + x_5) \]
\[(x_0, x_1, x_2) : (x_3 + b x_4 + x_5 + b x_6) = (x_0, x_1, x_2, x_4, x_5, x_3 + \frac{1}{a} x_6) \]
\[(x_0, x_1, x_2) : (x_3 + b x_4 + \frac{1}{a} x_6) = (x_0, x_1, x_2, x_3, x_4 + x_6) \]
\[(x_0, x_1, x_2) : (x_6) = (x_0, x_1, x_2, x_3, x_4, x_5, x_2 + x_6) \]
\[(x_0, x_2, x_4) : (x_6) = (x_0, x_1, x_2, x_5, x_3 + \frac{1}{a} x_4 + \frac{1}{a} x_6) \]
\[(x_0, x_1, x_2) : (x_6) = (x_0, x_1, x_2, x_5, x_3 + \frac{1}{a} x_4 + x_6) \]
\[(x_0, x_2, x_3) : (x_6) = (x_0, x_1, x_2, x_3, x_5 + \frac{1}{a} x_4 + \frac{1}{a} x_6) \]
\[(x_0, x_1, x_5) : (x_6) = (x_0, x_1, x_5, x_2 + a x_3 + x_4 + x_6) \]
\[(x_0, x_1, x_2) : (x_5) = (x_0, x_1, x_2, x_6, x_3 + b x_4 + \frac{1}{a} x_5) \]
\[(x_0, x_4, x_5) : (x_6) = (x_0, x_1, x_4, x_5, x_2 + a x_3 + \frac{1}{a} x_5) \]
\[(x_0, x_1, x_5) : (x_2 + a x_3 + x_4 + x_6) = (x_0, x_1, x_5, x_3 + b x_4) \]
\[(x_0, x_1, x_2, x_5) : (x_3 + b x_4 + b x_6) = (x_0, x_1, x_2, x_4, x_5, x_3 + \frac{1}{a} x_6) \]
\[(x_0, x_2, x_4, x_6) : (x_1 + x_3 + \frac{1}{a} x_5) = (x_0, x_2, x_3, x_4, x_5 + \frac{1}{a} x_6) \]
\[(x_0, x_2, x_3, x_6) : (x_4 + \frac{1}{b} x_5) = (x_0, x_2, x_4, x_6, x_1 + \frac{1}{b} x_5) \]
\[(x_0, x_3, x_4, x_6) : (x_5) = (x_0, x_2, x_3, x_4, x_6, x_1 + \frac{1}{b} x_5) \]
\[(x_0, x_1, x_2, x_6) : (x_3 + b x_4 + \frac{1}{a} x_5) = m_R \]
\[(x_0, x_1, x_2, x_5) : (x_4) = (x_0, x_1, x_2, x_5, x_3 + b x_4 + b x_6) \]
\[(x_0, x_2, x_3, x_6) : (x_4) = (x_0, x_1, x_2, x_3, x_6, x_4 + \frac{1}{b} x_5) \]
\[(x_0, x_1, x_2, x_5) : (x_3) = (x_0, x_1, x_2, x_5, x_3 + b x_4 + \frac{1}{a} x_6) \]
\[(x_0, x_1, x_2, x_5) : (x_3 + \frac{1}{a} x_4 + \frac{1}{a} x_6) = (x_0, x_1, x_2, x_5, x_6, x_3 + b x_4) \]
\[(x_0, x_1, x_2, x_5) : (x_3 + b x_4 + \frac{1}{a} x_6) = (x_0, x_1, x_2, x_3, x_5, x_4 + x_6) \]
(x_0, x_1, x_4, x_5): (x_3) = (x_0, x_1, x_4, x_5, x_2 + a \bar{x}_3 + \bar{x}_6)

(x_0, x_1, x_2, x_6): (x_3 + b \bar{x}_4) = (x_0, x_1, x_2, x_6, x_3 + b \bar{x}_4 + \bar{x}_5)

(x_0, x_1, x_4, x_5): (x_2 + a \bar{x}_3 + \bar{x}_6) = (x_0, x_1, x_3, x_4, x_5, x_6)

(x_0, x_1, x_5, x_6): (x_3 + b \bar{x}_4) = (x_0, x_1, x_4, x_5, x_6, x_2 + a \bar{x}_3)

(x_0, x_1, x_4, x_5): (\bar{x}_6) = (x_0, x_1, x_4, x_5, x_2 + a \bar{x}_3 + \bar{x}_6)

(x_0, x_1, x_2, x_5): (\bar{x}_6) = (x_0, x_1, x_2, x_5, x_3 + \frac{1}{a} \bar{x}_4 + \frac{1}{a} \bar{x}_6)

(x_0, x_1, x_2, x_6): (x_4) = (x_0, x_1, x_2, x_6, x_3 + b \bar{x}_4 + \bar{x}_5)

(x_0, x_1, x_2, x_6): (x_3 + b \bar{x}_4 + \bar{x}_5) = m_R

(x_0, x_1, x_2, x_4, x_5): (x_3 + \frac{1}{a} \bar{x}_6) = m_R

(x_0, x_1, x_2, x_3, x_5): (x_4 + \bar{x}_6) = m_R

(x_0, x_1, x_3, x_4, x_5): (x_2 + \bar{x}_6) = (x_0, x_1, x_3, x_4, x_5, x_6)

(x_0, x_3, x_4, x_5, x_6): (\bar{x}_1) = (x_0, x_3, x_4, x_5, x_6)

(x_0, x_3, x_4, x_5, x_6): (\bar{x}_2) = (x_0, x_3, x_4, x_5, x_6)

(x_0, x_2, x_3, x_4, x_6): (\bar{x}_1 + \bar{x}_5) = (x_0, x_2, x_3, x_4, x_5, x_6)

(x_0, x_2, x_3, x_6, x_4 + \frac{1}{a} x_5): (\bar{x}_1) = (x_0, x_2, x_3, x_4, x_5, x_6)

(x_0, x_1, x_4, x_5, x_2 + a \bar{x}_3 + \bar{x}_6): (\bar{x}_6) = (x_0, x_1, x_4, x_5, x_2 + a \bar{x}_3 + \bar{x}_6)

(x_0, x_1, x_5, x_6, x_3 + b \bar{x}_4): (\bar{x}_2) = (x_0, x_1, x_3, x_4, x_5, x_6)

(x_0, x_1, x_2, x_4, x_6): (\bar{x}_3 + \bar{x}_5) = m_R

(x_0, x_1, x_3, x_4, x_5, x_6): (\bar{x}_2) = (x_0, x_1, x_3, x_4, x_5, x_6)

Note that |F| = 57. Every colon is a non-trivial calculation and requires significant effort; the interested reader may find a link to Macaulay2 code verifying the filtration in the Appendix.
4.6 Hilbert Function Obstructions to the Koszul Property

In this section, we determine when the coordinate ring of a generic collection of lines is not Koszul. But first, we need a theorem from Complex Analysis.

Theorem 4.6.1. (Remmert, 1991, Vivanti–Pringsheim Theorem) Let the power series \( f(z) = \sum a_v z^v \) have positive finite radius of convergence \( r \) and suppose that all but finitely many of its coefficients \( a_v \) are real and non-negative. Then \( z = r \) is a singular point of \( f(z) \).

Theorem 4.6.2. Let \( \mathcal{M} \) be a generic collection of \( m \) lines in \( \mathbb{P}^n \) with \( n \geq 2 \) and \( R \) the coordinate ring of \( \mathcal{M} \). If
\[
m > \frac{1}{72} \left( 3(n^2 + 10n + 13) + \sqrt{3(n-1)^3(3n+5)} \right),
\]
then \( R \) is not Koszul.

Proof. We prove the claim by contradiction. Suppose that \( \text{reg}_S(R) = \alpha \). Note that by Theorem 4.4.5, \( \alpha \) is the smallest non-negative integer such that \( \binom{n+\alpha}{\alpha} \geq m(\alpha+1) \). We have four cases: \( \alpha = 0 \), \( \alpha = 1 \), \( \alpha = 2 \), or \( \alpha \geq 3 \).

1. Suppose that \( \alpha = 0 \). Then
\[
1 < \frac{1}{72} \left( 3(n^2 + 10n + 13) + \sqrt{3(n-1)^3(3n+5)} \right) < m \leq 1,
\]
a contradiction.

2. If \( \alpha = 1 \), then \( 2m \leq n + 1 \), and hence
\[
m \leq \frac{n + 1}{2} < \frac{1}{72} \left( 3(n^2 + 10n + 13) + \sqrt{3(n-1)^3(3n+5)} \right) < m,
\]
a contradiction.

3. Now assume that \( \alpha = 2 \) and that \( R \) is Koszul. By Lemma 4.5.2, the Hilbert series for \( R \) is
\[
H_R(t) = \frac{(n+1-2m)t^3 + (3m-2n-1)t^2 + (n-1)t + 1}{(1-t)^2}.
\]
Thus, by Equation (4.2)

\[ P_R^C(t) = \frac{1}{H_R(-t)} = \frac{(1+t)^2}{(2m-n-1)t^4+(3m-2n-1)t^2+(1-n)t+1}. \]

Denote

\[ p(t) = 1 + (1-n)t + (3m-2n-1)t^2 + (2m-n-1)t^3 \]

and note the leading coefficient is positive, since \( n + 1 < 2m \). By the Intermediate Value Theorem \( p(t) \) has a negative zero, since \( p(0) = 1 \) and

\[ p(-3) = -27m + 12n + 16 < 0, \]

since \( n + 1 < 2m \) and \( 1 < m \). So, the radius of convergence \( r \) of \( P_R^C(t) \) is finite and all the coefficients are positive. So, by Theorem 4.6.1, \( r \) must occur as a singular point of \( P_R^C(t) \); meaning that \( p(t) \) must have 3 real roots and one of them must be positive. Recall that if the discriminant of a cubic polynomial with real coefficients is negative, then the polynomial has 2 non-real complex roots. Thus, the discriminant of \( p(t) \) must be non-negative. The discriminant of \( p(t) \) is

\[ \Delta = -m(108m^2 - 9m(n^2 + 10n + 13) + 4(n + 2)^3). \]

We view the discriminate as a continuous function of \( m \). Now, note that the leading term of \( \Delta \) is negative. Applying the quadratic formula to the quadratic term above and only considering the larger root of the two yields the following

\[
m = \frac{9(n^2 + 10n + 13) + \sqrt{9^2(n^2 + 10n + 13)^2 - 4(108)(4)(n + 2)^3}}{2(108)}
\]

\[
= \frac{3(n^2 + 10n + 13) + \sqrt{9n^4 - 12n^3 - 18n^2 + 36n - 15}}{72}
\]

\[
= \frac{3(n^2 + 10n + 13) + \sqrt{3(n-1)^2(3n+5)}}{72}.
\]

Since, we have a unique positive root in the quadratic term and \( m > 0 \), we may conclude that

\[ m \leq \frac{1}{72} \left( 3(n^2 + 10n + 13) + \sqrt{3(n-1)^2(3n+5)} \right), \]

a contradiction.
4. Suppose that $\alpha \geq 3$ and $R$ is Koszul. By Theorem 4.4.4, the defining ideal of $R$ contains
a form of degree $\alpha$ in a minimal generating set, where $\alpha \geq 3$. Thus, $R$ is not quadratic, a
contradiction.

Hence, $R$ is not Koszul.

We have at least one exceptional example of a coordinate ring of a generic collection of lines
that is not Koszul that the previous theorem does not handle.

**Proposition 4.6.3.** Let $\mathcal{M}$ be a collection of 3 lines in general linear position in $\mathbb{P}^4$ and $R$ the
coordinate ring of $\mathcal{M}$. The defining ideal $J$ for $R$ has a cubic in a minimal generating set. Hence,
$R$ is not Koszul.

**Proof.** By Remark 4.3.8 and a change of basis, we may assume the defining ideals for our three
lines have the form

$$L_1 = (x_0, x_1, x_3), \quad L_2 = (x_0, x_2, x_4), \quad L_3 = (x_1, x_2, l),$$

where $l = x_3 + x_4$. Let $J$ be the defining ideal for $R$ and notice that

$$K = L_1 \cap L_2 = (x_0, x_1 x_2, x_1 x_4, x_3 x_2, x_3 x_4).$$

We have the following ring isomorphism

$$S/(K + L_3) = C[x_0, x_1, x_2, x_3, x_4]/(x_0, x_1, x_2, l, x_3 x_4)$$

$$\cong C[x_3, x_4]/(l, x_3 x_4)$$

$$\cong C[w]/(w^2).$$

Hence,

$$H_{S/(K+L_3)}(t) = \frac{-t^2 + 1}{1 - t} = 1 + t.$$

Therefore, by Proposition 4.3.2 the $\text{reg}_S(S/(K + L_3)) = 1$.

One checks that the $\text{reg}_S(S/K) = 1$. Using the short exact sequence

$$0 \to S/J \to S/K \oplus S/L_3 \to S/(K + L_3) \to 0$$
and Proposition 4.3.2 yields \( \text{reg}(S/J) \leq 2 \). So \( J \) is generated by forms of degree at most 3. The previous short exact sequence, Lemma 4.5.2, and the additivity of the Hilbert series along the previous short exact sequence yields

\[
H_{S/J}(t) = H_{S/K}(t) + H_{S/L_3}(t) - H_{S/(K+L_3)}(t)
\]

\[
= -\frac{t^2 + 2t + 1}{(1-t)^2} + \frac{1}{(1-t)^2} - (1 + t)
\]

\[
= -t^3 + 3t + 1
\]

\[
= 1 + 5t + 9t^2 + 12t^3 + \ldots.
\]

Thus, \( J \) is generated by 6 linearly independent quadrics and possibly cubics. The cubic \( x_3x_4 \) is contained in \( J \), but is not contained in the ideal \((x_0x_1, x_0x_2, x_1x_4, x_2x_3, x_0l)\), since no term divides \( x_3^2x_4 \). Hence, there must be a cubic generator in a minimal generating set of \( J \). Thus, \( R \) is not Koszul.

\[\square\]

**Remark 4.6.4.** Since Remark 4.3.8 says that a generic collection of lines is in general linear position, then we may use Lemma 4.5.2 to show that the coordinate ring of a generic collection of 3 lines in \( \mathbb{P}^4 \) has the same Hilbert series as \( R \).

### 4.7 Examples

Finally it is worth observing 3 examples that have appeared while studying generic lines.

**Example 4.7.1.** There are collections of lines in general linear position that are not generic collections. Consider the four lines in \( \mathbb{P}^3 \):

\[
\mathcal{L}_1 = \{[0 : 0 : \alpha : \beta] : \alpha, \beta \text{ not both zero}\}
\]

\[
\mathcal{L}_2 = \{[\alpha : \beta : 0 : 0] : \alpha, \beta \text{ not both zero}\},
\]

\[
\mathcal{L}_3 = \{[\alpha : \beta : -\alpha : \beta] : \alpha, \beta \text{ not both zero}\},
\]

\[
\mathcal{L}_4 = \{[\alpha : -\beta : \alpha : \beta] : \alpha, \beta \text{ not both zero}\}.
\]
These lines are in general linear position since every pair spans $\mathbb{P}^3$. The four defining ideals in $S$ are

\begin{align*}
L_1 &= (x_0, x_1), \\
L_2 &= (x_2, x_3), \\
L_3 &= (x_0 + x_2, x_1 - x_3), \\
L_4 &= (x_0 - x_2, x_1 + x_3).
\end{align*}

The coordinate ring $S/J$, where $J = \bigcap_{i=1}^4 L_i$, has the following Hilbert series

\[ H_{S/J}(t) = \frac{-3t^4 + 2t^3 + 2t^2 + 2t + 1}{(1 - t)^2} = 1 + 4t + 9t^2 + \cdots, \]

whereas, by Theorem 4.4.4, the coordinate ring $R$ for 4 generic lines in $\mathbb{P}^3$ has the following Hilbert series:

\[ H_R(t) = \frac{-2t^3 + 3t^2 + 2t + 1}{(1 - t)^2} = 1 + 4t + 10t^2 + \cdots. \]

So, this is not a generic collection of lines.

**Example 4.7.2.** Consider the coordinate ring $R$ for 5 generic lines in $\mathbb{P}^5$. The defining ideal $J$ for $R$ is minimally generated by quadrics and has the following Betti table computed via Macaulay2.

```
<table>
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<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>-</td>
<td>6</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>-</td>
<td>-</td>
<td>25</td>
<td>36</td>
<td>20</td>
<td>4</td>
</tr>
</tbody>
</table>
```

The ring $R$ is not Koszul by Theorem 4.6.2. Furthermore, it is known that if $R$ is Koszul and the defining ideal is generated by $g$ elements, then $\beta_{i,2i} \leq \binom{g}{i}$ for $i \in \{2, \ldots, g\}$ Avramov et al. (2010). The previous inequality fails for $i = 2$. So, this ring is not Koszul for two numerical reasons.

**Example 4.7.3.** Consider the coordinate ring $R$ for 6 generic lines in $\mathbb{P}^6$. The defining ideal $J$ for $R$ is minimally generated by quadrics and has the following Betti table computed via Macaulay2.
The Algebra is not Koszul by Theorem 4.6.2, and does not fail the aforementioned inequality.

Coordinate rings with defining ideals minimally generated by quadrics are not rare, but the previous two examples are interesting since both fail for identical reasons, and one fails for an additional numerical reason. It would be interesting to determine sufficient reasons why certain numerical conditions fail and others do not. For example, why does $\beta_{i,2i} \leq \binom{g}{i}$ fail in one of the previous rings but not the other.

Furthermore, we would like to add that our theorems do not cover every coordinate ring $R$ for every generic collection of lines in $\mathbb{P}^n$. For the coordinate rings we could not determine, there is a possibility these rings could be LG-quadratic or G-quadratic. In every possible case computable by Macaulay2 there exists a quadratic monomial ideal whose quotient ring gives the same Hilbert series as $R$. There could even be some change of basis which gives a quadratic Gröbner basis. Further, if we wanted to construct a Koszul filtration in these coordinate rings, then Proposition 4.5.10 demonstrates that there is no reason we should expect a reasonable filtration unless there is a more efficient change of basis that went unobserved. Below is a table, without $m = 1$ and $n = 1$ and $n = 2$, summarizing our results:
The Koszul property for the coordinate ring of $m$ generic lines in $\mathbb{P}^n$

4.8 Appendix

The following is Macaulay2 code to verify that the Koszul filtration in Proposition 4.5.10 is valid. The first line of the code generates a ring $T$ in two variables. The second line generates the field of fractions $F$ of $T$. The third line generates the ring $S$ with coefficients from $F$. The remaining code generates the defining ideal for our collection of lines and verifies the colon computations guaranteeing that this is a Koszul filtration.

\[ T = \mathbb{Z}/32003[a,b]; \]
\[ F = \text{frac}(T); \]
\[ S = F[x_0..x_6]; \]
\[ \text{L1 = ideal}(x_0,x_3,x_4,x_5,x_6); \]
\[ \text{L2 = ideal}(x_0,x_1,x_4,x_5,x_2+a*x_3+x_6); \]
\[ \text{L3 = ideal}(x_0,x_1,x_2,x_6,x_3+b*x_4+x_5); \]
\[ \text{L4 = ideal}(x_1,x_2,x_3,x_5,x_0+x_4+x_6); \]
\[ \text{L5 = ideal}(x_2,x_3,x_4,x_6,x_0+x_1+x_5); \]
\[ J = \text{intersect}(\text{L1, L2, L3, L4, L5}); \]
\[ R = S/J; \]

\[
\begin{align*}
\text{ideal}(0_R):\text{ideal}(x_0) &= \text{ideal}(x_2,x_3,x_0+x_1+x_4+x_5+x_6) \\
\text{ideal}(0_R):\text{ideal}(x_2) &= \text{ideal}(x_0,x_4,x_5) \\
\text{ideal}(x_0):\text{ideal}(x_4) &= \text{ideal}(x_0,x_1,x_2,x_3+b*x_4+x_5+b*x_6) \\
\text{ideal}(x_0):\text{ideal}(x_1) &= \text{ideal}(x_0,x_3,x_4,x_6) \\
\text{ideal}(x_2):\text{ideal}(x_3) &= \text{ideal}(x_0,x_1,x_2,x_3+b*x_4+x_5+(1/a)*x_6) \\
\text{ideal}(x_0):\text{ideal}(x_6) &= \text{ideal}(x_0,x_1,x_5,x_2+a*x_3+x_4+x_6) \\
\text{ideal}(x_2,x_3):\text{ideal}(x_0) &= \text{ideal}(x_2,x_3,x_0+x_1+x_4+x_5+x_6) \\
\text{ideal}(x_2,x_3):\text{ideal}(x_0+x_1+x_4+x_5+x_6) &= \text{ideal}(x_0,x_2,x_3,x_6,x_4+(1/b)*x_5) \\
\text{ideal}(x_0,x_4):\text{ideal}(x_5) &= \text{ideal}(x_0,x_2,x_4,x_6,x_1+x_3+x_5) \\
\text{ideal}(x_0,x_1):\text{ideal}(x_2) &= \text{ideal}(x_0,x_1,x_4,x_5) \\
\text{ideal}(x_0,x_4):\text{ideal}(x_3) &= \text{ideal}(x_0,x_1,x_4,x_2+a*x_3+a*x_5+x_6) \\
\text{ideal}(x_0,x_1):\text{ideal}(x_4) &= \text{ideal}(x_0,x_1,x_2,x_3+b*x_4+x_5+b*x_6) \\
\text{ideal}(x_0,x_4):\text{ideal}(x_2) &= \text{ideal}(x_0,x_4,x_5) \\
\text{ideal}(x_0,x_6):\text{ideal}(x_2) &= \text{ideal}(x_0,x_4,x_5,x_6) \\
\text{ideal}(x_0,x_1):\text{ideal}(x_5) &= \text{ideal}(x_0,x_1,x_2,x_6,x_3+b*x_4+x_5) \\
\text{ideal}(x_0,x_1,x_2):\text{ideal}(x_3+b*x_4+x_5+b*x_6) &= \text{ideal}(x_0,x_1,x_2,x_4,x_5,x_3+(1/a)*x_6) \\
\text{ideal}(x_0,x_1,x_2):\text{ideal}(x_3+b*x_4+x_5+(1/a)*x_6) &= \text{ideal}(x_0,x_1,x_2,x_3,x_5,x_4+x_6) \\
\text{ideal}(x_0,x_3,x_4):\text{ideal}(x_6) &= \text{ideal}(x_0,x_1,x_3,x_4,x_5,x_2+x_6) 
\end{align*}
\]
ideal(x_0,x_1,x_4):ideal(x_5) == ideal(x_0,x_1,x_2,x_4,x_6,x_3+x_5)
ideal(x_0,x_1,x_4):ideal(x_2+a*x_3+a*x_5+x_6) == ideal(x_0,x_1,x_3,x_4,x_5,x_6)
ideal(x_0,x_2,x_4):ideal(x_6) == ideal(x_0,x_1,x_2,x_4,x_5,x_3+(1/a)*x_6)
ideal(x_0,x_1,x_2):ideal(x_6) == ideal(x_0,x_1,x_2,x_5,x_3+(1/a)*x_4+(1/a)*x_6)
ideal(x_0,x_2,x_3):ideal(x_6) == ideal(x_0,x_1,x_2,x_3,x_5,x_4+x_6)
ideal(x_0,x_1,x_5):ideal(x_6) == ideal(x_0,x_1,x_5,x_2+a*x_3+x_4+x_6)
ideal(x_0,x_1,x_2):ideal(x_5) == ideal(x_0,x_1,x_2,x_6,x_3+b*x_4+x_5)
ideal(x_0,x_4,x_5):ideal(x_6) == ideal(x_0,x_1,x_4,x_5,x_2+a*x_3+x_6)
ideal(x_0,x_1,x_5):ideal(x_2+a*x_3+x_4+x_6) == ideal(x_0,x_1,x_5,x_6,x_3+b*x_4)
ideal(x_0,x_1,x_2,x_5):ideal(x_6) == ideal(x_0,x_1,x_2,x_5,x_3+(1/a)*x_4+(1/a)*x_6)
ideal(x_0,x_1,x_2,x_6):ideal(x_3+b*x_4+x_5) == ideal(x_0,x_1,x_2,x_6,x_3+b*x_4+x_5)
ideal(x_0,x_1,x_4,x_5):ideal(x_3 ) == ideal(x_0,x_1,x_4,x_5,x_2+a*x_3+x_6)
ideal(x_0,x_1,x_2,x_6):ideal(x_3+b*x_4) == ideal(x_0,x_1,x_2,x_6,x_3+b*x_4+x_5)
ideal(x_0,x_1,x_4,x_5):ideal(x_2+a*x_3+x_6) == ideal(x_0,x_1,x_3,x_4,x_5,x_6)
ideal(x_0,x_1,x_2,x_4,x_5):ideal(x_3+(1/a)*x_6) == ideal(x_0..x_6)
ideal(x_0,x_1,x_2,x_3,x_5):ideal(x_4+x_6) == ideal(x_0..x_6)
ideal(x_0,x_1,x_2,x_3,x_4,x_5):ideal(x_2+x_6) == ideal(x_0,x_1,x_3,x_4,x_5,x_6)
ideal(x_0,x_2,x_3,x_4,x_5,x_6):ideal(x_1) == ideal(x_0,x_1,x_3,x_4,x_5,x_6)
ideal(x_0,x_2,x_3,x_4,x_5,x_6):ideal(x_2) == ideal(x_0,x_1,x_3,x_4,x_5,x_6)
ideal(x_0,x_2,x_3,x_4,x_5,x_6):ideal(x_1+x_5) == ideal(x_0,x_2,x_3,x_4,x_5,x_6)
ideal(x_0,x_2,x_3,x_4,x_5,x_6):ideal(x_2+a*x_3+x_6) == ideal(x_0,x_2,x_3,x_4,x_5,x_6)
ideal(x_0,x_2,x_3,x_4,x_5,x_6):ideal(x_3+b*x_4) == ideal(x_0,x_1,x_3,x_4,x_5,x_6)
ideal(x_0,x_1,x_2,x_4,x_6):ideal(x_3+x_5) == ideal(x_0..x_6)
ideal(x_0,x_1,x_3,x_4,x_5,x_6):ideal(x_2) == ideal(x_0,x_1,x_3,x_4,x_5,x_6)

4.9 References


Koszul algebras and their related numerical and homological properties have come under more intense study in recent years. More research has been done on classes of algebras that admit the Koszul property. In particular, more work has been done on different characterizations of the Koszul property regarding their various properties. In Chapter 3, we studied various conditions necessary for an algebra to be Koszul and determined that several are independent. We have only determined a partial list of necessary numerical conditions for Koszul algebras which are independent of one another.

Finally, in Chapter 4, we studied the coordinate ring $R$ of a generic collection $M$ of $m$ lines in $\mathbb{P}^n$. We determined that if

\[
m > \frac{1}{72} \left( 3(n^2 + 10n + 13) + \sqrt{3(n - 1)^3(3n + 5)} \right),
\]

then $R$ is not Koszul. Furthermore, we determined that:

(a) If $m$ is even and $m + 1 \leq n$, then $R$ has a Koszul filtration.

(b) If $m$ is odd and $m + 2 \leq n$, then $R$ has a Koszul filtration.

It was observed that a generic collection of 3 lines in $\mathbb{P}^4$ was an exceptional algebra that did not admit the Koszul property and that a generic collection of 5 lines in $\mathbb{P}^6$ was an exceptional algebra that did admit the Koszul property. In the future, it would be interesting to completely determine if every coordinate ring of $m$ lines in $\mathbb{P}^n$ such that

\[
n \leq m \leq \frac{1}{72} \left( 3(n^2 + 10n + 13) + \sqrt{3(n - 1)^3(3n + 5)} \right),
\]

admits the Koszul property. Computationally, these algebras do not seem to admit the G-quadratic or LG-quadratic properties. Thus, determining if these algebras admit these properties would be an interesting future project.