Linear Syzygies of Toric Edge Ideals of Bipartite Graphs

Jason McCullough (joint with Zach Greif)

Iowa State University

JMM Special Session on Commutative Algebra

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Outline

1. Notation
2. Regularity vs. Linear Syzygies
3. Toric Edge Ideals
4. Main Result
5. A Regularity Result
6. Polyominoes and Hibi Rings

Notation

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Green-Lazarsfeld Index

Definition

The ideal I satisfies property N_p if S/I is normal and $\beta_{ij} = 0$ for $i \leq p$ and $j > 2 + i$.

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Property N_1: projectively normal + generated by quadrics
Property N_2: projectively normal + generated by quadrics + linear first syzygies

Motivating Question

Question (Constantinescu-Kahle-Varbaro 2016)

Is there a family of quadratically generated ideals $\{I_n \subseteq R = k[x_1, \ldots, x_n]\}_{n \in \mathbb{N}}$ with linear first syzygies (N_2) such that

$$\lim_{n \to \infty} \frac{\operatorname{reg}(I_n)}{n} > 0?$$

Monomial Ideals

Theorem (Constantinescu-Kahle-Varbaro)

For any integers $r, s \geq 0$, there exists a squarefree, quadratic monomial ideal with linear syzygies for r steps (N_r) and with regularity $\geq s$.

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Construction uses lots of variables...

Monomial Ideals

Theorem (Dao-Huneke-Schweig)

Let G be a graph such that $I(G)$ is k-steps linear for some $k \geq 1$. Then

$$\operatorname{reg}(I(G)) \leq \log_{\frac{k+4}{2}}\left(\frac{n-1}{k+1}\right) + 3.$$

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What about binomial ideals?

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$G = (V, E)$ a simple graph
$V = \{v_1, \ldots, v_n\}$
$E = \{e_1, \ldots, e_r\}$ where $e_i = \{v_{i_1}, v_{i_2}\}$

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$K[G] := K[v_{i_1} v_{i_2}] \cong K[e_1, \ldots, e_r]/I_G$

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Generated by binomials associated to even closed walks in G

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Toric Edge Ideals

Theorem (Ohsugi-Hibi)

I_G has a 2-linear resolution ($N_p \forall p$) if and only if G is a complete bipartite graph $K_{2,n}$. ($\mathbb{P}^1 \times \mathbb{P}^{n-1}$)

Bipartite Graphs

Theorem (Ohsugi-Hibi)

Let G be a bipartite graph. The following are equivalent:

- *Every cycle in G of length ≥ 6 has a chord. ("chordal")*
- *I_G has a Gröbner basis consisting of quadratic binomials.*
- *$K[G]$ is Koszul.*
- *I_G is generated by quadratic binomials, corresponding to the even cycles of G.*
- *I_G satisfies property N_1*

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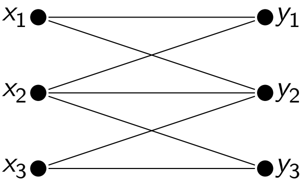
Question

When is $\beta_{1,j}(I_G) = 0$ for $j > 3$? i.e. When does I_G satisfy N_2?

Two Examples



$$I_G = (z_{11}z_{22} - z_{12}z_{21}, z_{22}z_{33} - z_{23}z_{32}) \subseteq K[z_{ij}]$$

Betti table:

	0	1
2:	2	-
3:	-	1

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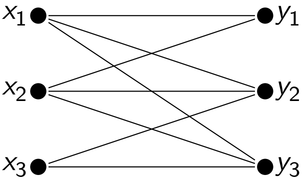
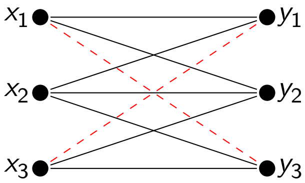


Betti table:

	0	1	
2:	5	5	-
3:	-	-	1

Global Criteria for Linear Syzygies

G a connected, bipartite graph with vertex set $X \sqcup Y$ with degree of every vertex ≥ 2.
The **bipartite complement** of $G = (X \sqcup Y, E)$, denoted $\overline{G}$, is the bipartite graph with edge set $(X \times Y) \setminus E$.

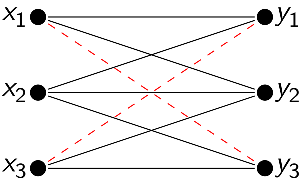


Global Criteria for Linear Syzygies

Theorem (Greif,-)

Let $G = (V, E)$ be a connected, chordal bipartite graph with $\deg(v) \geq 2$ for all $v \in V$ and let K be a field. Then I_G has satisfies property N_2 if and only if $\overline{G}$ is a tree of diameter at most 3.

Re: Examples



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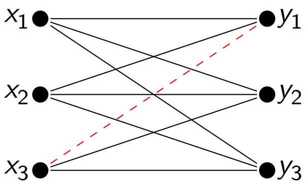

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	0	1	
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3:	-	-	1

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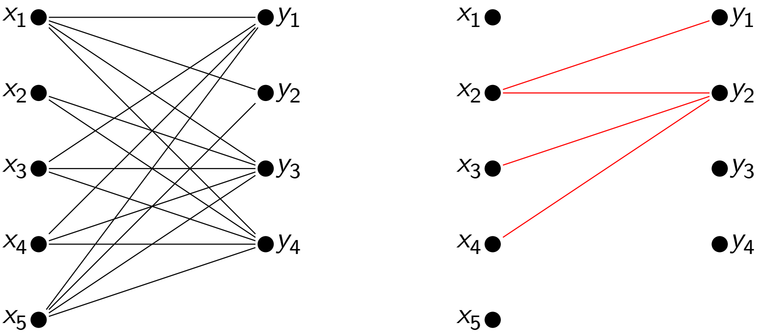
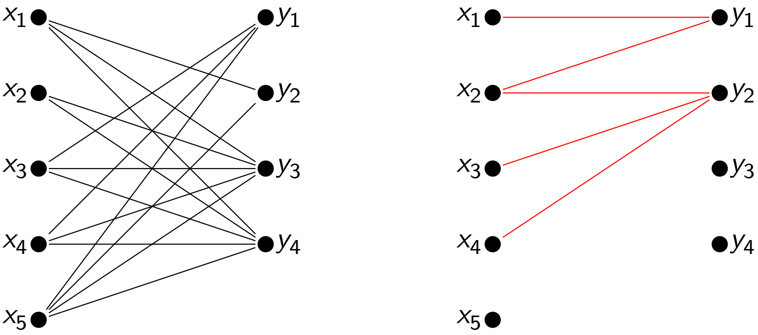


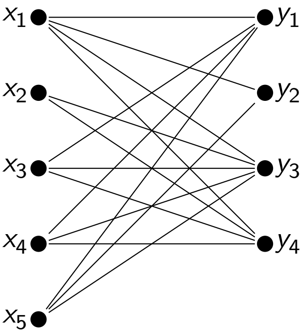
	0	1	2	3	4	5	6	7
2:	25	83	94	34	-	-	-	-
3:	-	-	62	182	190	77	6	-
4:	-	-	-	-	-	10	13	3

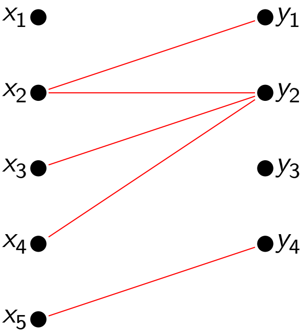
Re: Examples



	0	1	2	3	4	5	6
2:	18	45	30	4	-	-	-
3:	-	9	67	105	57	6	-
4:	-	-	-	-	6	11	3

Re: Examples





	0	1	2	3	4	5	6
2:	17	38	21	-	-	-	-
3:	-	21	98	140	78	13	1
4:	-	-	-	-	6	11	3

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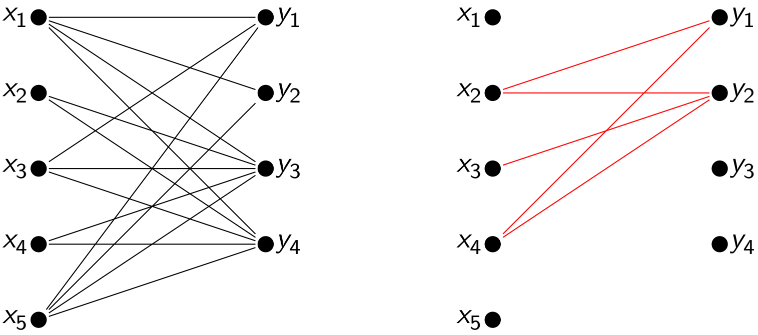

	0	1	2	3	4	5	6
2:	19	51	36	4	-	-	-
3:	-	1	53	105	71	14	-
4:	-	-	-	-	-	5	2

Classification of N_p Property

Theorem

Let $G = (X \sqcup Y, E)$ be a chordal bipartite graph such that $\min_{v \in X \sqcup Y} \deg(v) \geq 2$.

- *I_G satisfies $N_1 \Leftrightarrow G$ is chordal.*

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- *I_G satisfies $N_3 \Leftrightarrow G$ is a complete bipartite graph.*

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- *I_G satisfies $N_3 \Leftrightarrow G$ is a complete bipartite graph.*
- *I_G satisfies N_p for any/all $p \geq 4 \Leftrightarrow G$ is a complete bipartite graph $K_{2,n}$.*

Proof Sketch

Theorem

Let $G = (X \sqcup Y, E)$ be a chordal bipartite graph such that $\min_{v \in X \sqcup Y} \deg(v) \geq 2$. Then $\overline{G}$ is a tree of diameter at most 3 if and only if the following conditions hold:

- *Every 3-3 induced subgraph of $\overline{G}$ is edge connected.*
- *Every 4-4 induced subgraph of $\overline{G}$ has no cycles of length 4.*

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Use Hochster's formula to equate minimal quadratic syzygies with certain small forbidden induced subgraphs.

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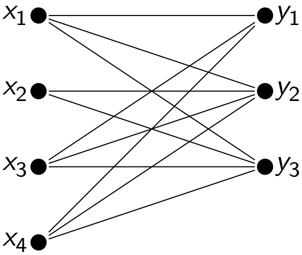
A Partial Answer

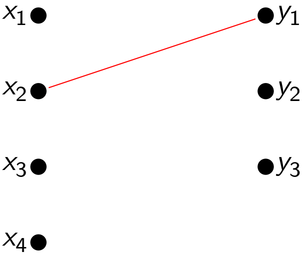
Corollary

For any family of graphs G_n, where G_n is bipartite with n edges and I_{G_n} has linear syzygies,

$$\lim_{n\to\infty} \frac{\operatorname{reg}(I_{G_n})}{n} = 0.$$

Nontriviality

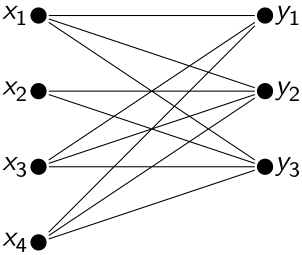


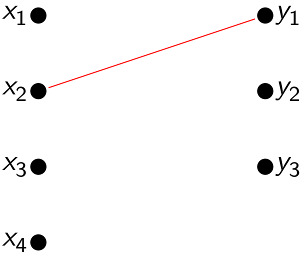


Betti table of I_G:

	0	1	2	3	4
1:	12	25	15	-	-
2 :	-	-	6	10	3

Nontriviality





Betti table of $LT(I_G)$:

	0	1	2	3	4
1:	12	25	19	6	1
2 :	-	4	12	11	3

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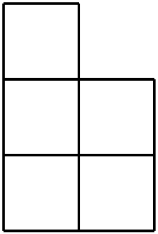
This is a polyomino:



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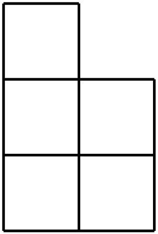

To each polyomino $\mathcal{P}$ associate the ideal:

$$I(\mathcal{P}) = (x_i y_j - x_k y_l \mid [(i,k),(j,l)] \subseteq \mathcal{P})$$

Polyomino Ideals

Theorem (Qureshi)

If $\mathcal{P}$ is convex, then $I(\mathcal{P})$ is the toric prime ideal associated to a chordal bipartite graph. In particular, $I(\mathcal{P})$ is generated by a quadratic Gröbner basis.

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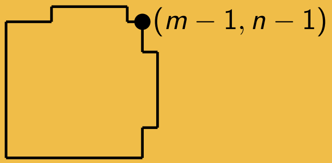
Theorem (Greif-M)

$I(\mathcal{P})$ is linearly related if and only if it has the following form:



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Let $\mathcal{L}$ be a finite distributive lattice.

What is a Hibi ring?

Let $\mathcal{L}$ be a finite distributive lattice. It's join-meet ideal

$$I(\mathcal{L}) = \left(x_\alpha x_\beta - x_{\alpha\wedge\beta} x_{\alpha\vee\beta} \,\middle|\, \alpha, \beta \in \mathcal{L}\right).$$

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This ideal presents the Hibi ring $K[\mathcal{L}]$ and so is prime and toric.

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This ideal presents the Hibi ring $K[\mathcal{L}]$ and so is prime and toric. If $\mathcal{L}$ is planar, $I(\mathcal{L})$ is a polyomino ideal!

Join-meet Ideals

Theorem (Birkhoff)

A finite lattice is distributive if and only if it is the poset $\mathcal{J}(P)$ of downsets of the poset P of join-irreducible elements of $\mathcal{L}$. It is planar if and only if $P = C \cup C'$ is the union of two chains.

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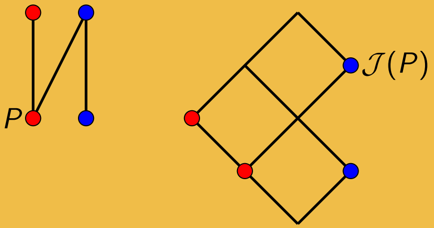
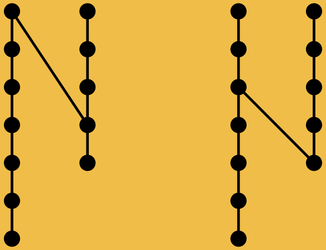
Example



Join-meet Ideals

Theorem (Ene, Greif-M)

Let $\mathcal{L} = \mathcal{J}(P)$ be a finite planar distributive lattice. Then $I(\mathcal{L})$ is linearly related if and only if the Hasse diagram of P has one of the following forms:



Thank you!

See: arXiv:1908.02744 [math.AC]